On Conjugacy of Second-Order Half-Linear Differential Equations on the Real Axis

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On the real axis, we consider the equation

$$\boxed{\big(|u'|^{\alpha}\operatorname{sgn} u'\big)' + p(t)|u|^{\alpha}\operatorname{sgn} u = 0,} \tag{1}$$

where $p:\mathbb{R}\to\mathbb{R}$ is a locally integrable function and $\alpha>0$.

A function $u: I\to\mathbb{R}$ is said to be *a solution to equation* (1) *on the interval* $I\subseteq\mathbb{R}$ if it is continuously differentiable on I, $|u'|^{\alpha}\operatorname{sgn} u'$ is absolutely continuous on every compact subinterval of I, and u satisfies equality (1) almost everywhere on I. In [6, Lemma 2.1], Mirzov proved that every solution to equation (1) is extendable to the whole real axis. Therefore, speaking about a solution to equation (1), we assume that it is defined on $\mathbb{R}$. Moreover, for any $a\in\mathbb{R}$, the initial value problem

$$\big(|u'|^{\alpha}\operatorname{sgn} u'\big)' + p(t)|u|^{\alpha}\operatorname{sgn} u = 0; \quad u(a)=0, \ \ u'(a)=0$$

has only the solution $u\equiv 0$ (see [6, Lemma 1.1]). Hence, a solution u to equation (1) is said to be *non-trivial*, if $u\not\equiv 0$ on $\mathbb{R}$.

Definition 1. We say that equation (1) is *conjugate on* $\mathbb{R}$ if it has a non-trivial solution with at least two zeros, and *disconjugate on* $\mathbb{R}$ otherwise.

It is clear that in the case $\alpha=1$, equation (1) reduces to the linear equation

$$u'' + p(t)u = 0. \tag{2}$$

As it is mentioned in [4], a history of the problem of conjugacy of (2) began in the paper by Hawking and Penrose [3]. In [8], Tipler presented an interesting relevance of the study of conjugacy of (2) to the general relativity and improved Hawking–Penrose's criterion, showing that (2) is conjugate on $\mathbb{R}$ if the inequality

$$\liminf_{\substack{t\to+\infty\\ \tau\to-\infty}} \int_{\tau}^{t} p(s)\,\mathrm{d}s > 0 \tag{3}$$

holds. Later, Peňa [7] proved that the same condition is sufficient also for the conjugacy of half-linear equation (1).

The study of conjugacy of (1) on $\mathbb{R}$ is closely related to the question of oscillation of (1) on the whole real axis. It is known that Sturms's separation theorem holds for equation (1) (see [6, Theorem 1.1]). Therefore, if equation (1) possesses a non-trivial solution with a sequence of zeros tending to $+\infty$ (resp. $-\infty$), then any other its non-trivial solution has also a sequence of zeros tending to $+\infty$ (resp. $-\infty$).

Definition 2. Equation (1) is said to be *oscillatory in the neighbourhood of* $+\infty$ (resp. *in the neighbourhood of* $-\infty$) if every its non-trivial solution has a sequence of zeros tending to $+\infty$ (resp. to $-\infty$). We say that equation (1) is *oscillatory on* $\mathbb{R}$ if it is oscillatory in the neighbourhood of either $+\infty$ or $-\infty$, and *non-oscillatory on* $\mathbb{R}$ otherwise.

Clearly, if equation (1) is oscillatory on $\mathbb{R}$, then it is conjugate on $\mathbb{R}$, as well. It is known that oscillations of (1) in the neighbourhood of $+\infty$ (resp. $-\infty$) can be described by means of behaviour of the Hartman–Wintner type expression

$$\frac{1}{|t|}\int_0^t\bigg(\int_0^s p(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s \tag{4}$$

in the neighbourhood of $+\infty$ (resp. $-\infty$), see [5, Theorem 12.3]. However, expression (4) is useful also in the study of conjugacy of (1) on $\mathbb{R}$. In particular, efficient conjugacy and disconjugacy criteria for linear equation (2) formulated by means of expression (4) are given in [4]. Abd-Alla and Abu-Risha [1] observed that for the study of conjugacy on whole real axis, it is more convenient to consider a Hartman–Wintner type expression in a certain symmetric form, where all values of the function p are involved simultaneously. They proved in [1], among other things, that equation (1) with a continuous p is conjugate on $\mathbb{R}$ provided that $p \not\equiv 0$ and

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t\bigg(\int_{-s}^s p(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s \geq 0, \tag{5}$$

which obviously improves Peňa's criterion (3). Below, we generalise and supplement criterion (5) and present further statements, which can be applied in the cases not covered by Theorems 3 and 5.

For any $\nu < 1$, we put

$$c(t;\nu) := \frac{1-\nu}{(1+t)^{1-\nu}}\int_0^t \frac{1}{(1+s)^\nu}\bigg(\int_{-s}^s p(\xi)\,\mathrm{d}\xi\bigg)\,\mathrm{d}s \ \text{ for } \ t \geq 0.$$

We start with a Hartman–Wintner type result, which guarantees that equation (1) is oscillatory on $\mathbb{R}$ (not only conjugate).

Theorem 3. *Let $\nu < 1$ be such that either*

$$\lim_{t\to+\infty} c(t;\nu) = +\infty,$$

or

$$-\infty < \liminf_{t\to+\infty} c(t;\nu) < \limsup_{t\to+\infty} c(t;\nu).$$

Then equation (1) *is oscillatory on* $\mathbb{R}$ *and consequently, conjugate on* $\mathbb{R}$.

Remark 4. Having $\nu_1, \nu_2 < 1$, one can show that there exists a finite limit $\lim\limits_{t\to+\infty} c(t;\nu_2)$ if and only if there exists a finite limit $\lim\limits_{t\to+\infty} c(t;\nu_1)$, in which case both limits are equal.

In view of Remark 4, Theorem 3 cannot be applied, in particular, if the function $c(\,\cdot\,;1-\alpha)$ has a finite limit as $t \to +\infty$. A conjugacy criterion covering this case is given in the following statement.

Theorem 5. *Let $p \not\equiv 0$ and*

$$0 \leq \lim_{t\to+\infty} c(t;1-\alpha) < +\infty.$$

Then equation (1) *is conjugate on* $\mathbb{R}$.

Theorems 3 and 5 yield

Corollary 6. *Let $p \not\equiv 0$ and $\nu < 1$ be such that*

$$\liminf_{t\to+\infty} c(t;\nu) > -\infty, \quad \limsup_{t\to+\infty} c(t;\nu) \geq 0.$$

Then equation (1) *is conjugate on* $\mathbb{R}$.

Corollary 6 generalises several conjugacy criteria known in the existing literature. In particular, [2, Theorem 2.2] can be derived from Corollary 6. Moreover, conjugacy criterion (5) given in [1, Theorem 2.2] follows immediately from Corollary 6 with $\nu := 0$. Corollary 6 also yields the following half-linear extension of [4, Theorem 1].

Corollary 7. *Let $p \not\equiv 0$ and the function*

$$M : t \longmapsto \frac{1}{|t|} \int\limits_0^t \bigg(\int\limits_0^s p(\xi)\,\mathrm{d}\xi \bigg)\,\mathrm{d}s$$

have finite limits as $t \to \pm\infty$. If

$$\lim_{t\to+\infty} M(t) + \lim_{t\to-\infty} M(t) \geq 0,$$

then equation (1) *is conjugate on* $\mathbb{R}$.

According to the above said, we conclude that neither of Theorems 3 and 5 can be applied in the following two cases:

$$\lim_{t\to+\infty} c(t;1-\alpha) =: c(+\infty) \in \,]-\infty, 0[\tag{6}$$

and

$$\liminf_{t\to+\infty} c(t;\nu) = -\infty \ \text{ for every } \ \nu < 1. \tag{7}$$

The case (6)

In the first statement, we require that the function $c(\,\cdot\,;1-\alpha)$ is at some point far enough from its limit $c(+\infty)$.

Theorem 8. *Let* (6) *hold and*

$$\sup \left\{ \frac{(1+t)^\alpha}{\ln(1+t)} \big[c(+\infty) - c(t;1-\alpha)\big] : \ t > 0 \right\} > 2\Big(\frac{\alpha}{1+\alpha}\Big)^{1+\alpha}. \tag{8}$$

Then equation (1) *is conjugate on* $\mathbb{R}$.

Remark 9. One can show that if (8) is replaced by

$$\limsup_{t\to+\infty} \frac{(1+t)^\alpha}{\ln(1+t)} \big[c(+\infty) - c(t;1-\alpha)\big] > 2\Big(\frac{\alpha}{1+\alpha}\Big)^{1+\alpha}, \tag{9}$$

then we can claim in Theorem 8 that equation (1) is even oscillatory on $\mathbb{R}$.

Now we put

$$Q_\alpha(t) := \frac{(1+t)^{1+\alpha}}{t} \bigg[c(+\infty) - \int\limits_{-t}^t p(s)\,\mathrm{d}s \bigg], \quad H_\alpha(t) := \frac{1}{t} \int\limits_{-t}^t (1+|s|)^{1+\alpha} p(s)\,\mathrm{d}s \ \text{ for } \ t > 0.$$

Theorem 10. *Let* (6) *hold and*

$$\sup\big\{Q_\alpha(t)+H_\alpha(t):\ t>0\big\}>2.$$

Then equation (1) *is conjugate on* $\mathbb{R}$.

Remark 11. One can show that if

$$\limsup_{t\to+\infty}\big(Q_\alpha(t)+H_\alpha(t)\big)>2,$$

then we can claim in Theorem 10 that equation (1) is even oscillatory on $\mathbb{R}$.

The case (7)

First note that, in condition (7), the assumption that $\liminf\limits_{\nu\to+\infty} c(t;\nu)=-\infty$ for **every** $\nu<1$ is, in fact, not too restrictive. Indeed, let $\liminf\limits_{t\to+\infty} c(t;\nu_1)=-\infty$ for some $\nu_1<1$. Then Remark 4 yields that for any $\nu<1$, the function $c(\,\cdot\,;\nu)$ does not possess any finite limit. Consequently, if there exists $\nu_2<1$ such that $\liminf\limits_{t\to+\infty} c(t;\nu_2)>-\infty$, then equation (1) is oscillatory on $\mathbb{R}$ as it follows from Theorem 3.

Proposition 12. *Let condition* (7) *hold and there exist a number* $\kappa>\alpha$ *such that*

$$\limsup_{t\to+\infty}\frac{1}{t^\kappa}\int\limits_{-t}^{t}(t-|s|)^\kappa p(s)\,\mathrm{d}s>-\infty. \tag{10}$$

Then equation (1) *is oscillatory on* $\mathbb{R}$ *and consequently, conjugate on* $\mathbb{R}$.

Finally, we give a statement which can be applied in the case, when condition (7) holds, but (10) is violated for every $\kappa>\alpha$, i.e.,

$$\lim_{t\to+\infty}\frac{1}{t^\kappa}\int\limits_{-t}^{t}(t-|s|)^\kappa p(s)\,\mathrm{d}s=-\infty\ \text{ for every }\ \kappa>\alpha$$

(it may happen as can be justified by an example).

Theorem 13. *Let there exist a number* $\kappa>\alpha$ *such that*

$$\sup\left\{\frac{1}{t^{\kappa-\alpha}}\int\limits_{-t}^{t}(t-|s|)^\kappa p(s)\,\mathrm{d}s:\ t>0\right\}>\frac{2}{\kappa-\alpha}\Big(\frac{\kappa}{1+\alpha}\Big)^{1+\alpha}.$$

Then equation (1) *is conjugate on* $\mathbb{R}$.

Remark 14. Observe that Theorem 13 does not require assumption (7), it is a general statement applicable without regard to behaviour of the function $c(\,\cdot\,;\nu)$.

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