# On the Solvability of One Class of Boundary Value Problems

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The questions of determining the conditions of solvability and finding the solutions for various types of boundary value problems remain actual for a long period of time. A vast number of scientific works are devoted to the investigation of different aspects of the question under consideration. The Noetherian boundary value problems have been considered and studied in [8]. The works [1] and [4] are devoted to the study of autonomous boundary value problems.

The weakly nonlinear boundary value problems have been considered in [1, 7]. The conditions for the solvability of boundary value problems with perturbation for systems of linear differential equations of the first order have been studied in [7, 8]. The conditions of solvability of degenerated boundary value problems, bifurcations and branching of their solutions are considered in [8]. In [6], the author considers weakly perturbed boundary value problems for systems of linear differential equations of the second order for which the conditions of solvability are found.

We study a linear inhomogeneous boundary value problem with perturbation

$$(P(t)x')' - Q(t)x - \varepsilon Q_1(t)x = f(t), \;\; t \in [a,b], \tag{1}$$

$$lx(\,\cdot\,,\varepsilon) = \alpha + \varepsilon l_1 x(\,\cdot\,,\varepsilon). \tag{2}$$

Here, $[a,b]$ is a segment on which we consider the linear boundary value problem with perturbations (1), (2), $x = x(t,\varepsilon)$ – is a twice continuously differentiable unknown vector-function $x''(\,\cdot\,,\varepsilon) \in C^2([a,b]\times(0,\varepsilon_0])$. $P(t)$, $Q(t)$, $Q_1(t)$ are square matrices of dimension $n$. Elements of the matrix $P(t)$ are real, continuously differentiable on the segment $[a,b]$ functions $P(t) \in C^1([a,b])$; Elements of the matrices $Q(t)$ and $Q_1(t)$ are continuous on the segment $[a,b]$: $Q(t), Q_1(t) \in C([a,b])$. The matrix $P(t)$ is nondegenerated $\det P(t) \neq 0$. The function $f(t)$ is a continuous $n$-dimensional on the segment $[a,b]$ vector-function $f(t) \in C([a,b])$. $l$, $l_1$ are linear bounded $m$-dimensional vector-functionals defined on the space $n$-dimensional piecewise continuous vector functions $l$, $l_1 : C([a,b]) \to R^m$. $\alpha$ is an $m$-dimensional real vector $\alpha \in R^m$; $\varepsilon$ is a small nonnegative parameter.

To the boundary value problem with perturbation (1), (2) we put into correspondence the generating boundary value problem

$$(P(t)x')' - Q(t)x = f(t), \;\; t \in [a,\ b], \tag{3}$$

$$lx(\,\cdot\,,\varepsilon) = \alpha. \tag{4}$$

The system (3) of differential equations of second order has a general solution of the type $x(t) = X(t)c + \overline{x}(t)$, $c \in R^{2n}$, where $X(t)$ is an $(n \times 2n)$-dimensions fundamental matrix of the homogeneous ($f(t) = 0$)system of second order (3) which consists of $2n$ linear independent solutions of that homogeneous system ($f(t) = 0$) (3); The vector-function $\overline{x}(t) = \int\limits_a^b K(t,s)P^{-1}(s)f(s)\,ds$ is a partial solution of the system of differential equations (3); $K(t,s)$ is the Cauchy $(n\times n)$-dimensional matrix [?, ?]. $D$ is a rectangula,$(m \times 2n)$-dimensional matrix formed under the action of the $m$-dimensional functional $l$ onto the fundamental matrix $X(t)$, $\operatorname{rank} D = n_1$, $n_1 < \min(2n,m)$. The matrix $D^*$ is transposed to the matrix $D$. The $(2n \times m)$-dimensional matrix $D^+$ is Moore–Penrose pseudo-inverse to the matrix $D$ [2, 5, 6, 8]. By $P_D$ we denote the $(2n \times 2n)$-dimensional matrix-orthoprojector $P_D : R^{2n} \to N(D)$, $N(D) = P_D R^{2n}$. The matrix $N(D)$ is the null-space of the matrix $D$: $\dim N(D) = 2n - \operatorname{rank} D = 2n - n_1 = r$. By $P_{D^*}$ we denoted the $(m \times m)$-measurable matrix-orthoprojector $P_{D^*} : R^m \to N(D^*)$, $N(D^*) = P_{D^*} R^m$. The matrix $N(D^*)$ is the null-space of the matrix $D^*$: $\dim N(D^*) = 2n - \operatorname{rank} D^* = 2n - n_1 = r$. Thus the matrix $N(D)$ is of dimension $r$: $\dim N(D) = 2n - \operatorname{rank} D = 2n - n_1 = r$, and the matrix $N(D^*)$ is of dimension $d$: $\dim N(D^*) = m - \operatorname{rank} D = m - n_1 = d$. Consequently, $\operatorname{rank} P_D = r$, $\operatorname{rank} P_{D^*} = d$, this implies that the matrix $P_D$ consists of $r$ linearly independent columns, and the matrix $P_{D^*}$ consists of $d$ linearly independent columns. Thus the $(2n \times 2n)$-dimensional matrix $P_D$ can be replaced by the $(2n \times r)$-dimensional matrix $P_{D_r}$ which consists of $r$ linearly independent columns of the matrix $P_D$; the $(m\times m)$-dimensional matrix $P_{D^*}$ can be replaced by $(d\times m)$-dimensional matrix $P_{D_d^*}$ which consists of $d$ linearly independent series of the matrix $P_{D^*}$ [3, 5].

For the generating boundary value problem (3), (4) the theorem below is fulfilled [5].

**Theorem 1** (Critical case)**.** *Let the condition* $\operatorname{rank} D = n_1 < \min\{2n, m\}$ *be fulfilled. Then the homogeneous* ($f(t) = 0$, $\alpha = 0$) *boundary value problem* (3), (4) *has* $r$, ($r = 2n - n_1$) *and only* $r$ *linearly independent solutions. The inhomogeneous boundary value problem* (3), (4) *is solvable if and only if the vector-function* $f(t) \in C([a, b])$ *and the constant vector* $\alpha \in R^m$ *satisfy the condition of solvability*

$$P_{D_d^*}\big[\alpha - l\overline{x}(\,\cdot\,)\big] = 0 \ \ (d = m - n_1). \tag{5}$$

*If these conditions are fulfilled, the boundary value problem* (3), (4) *has an* $r$*-parametric set of solutions* $x(t, c_r) = X_r(t)c_r + (G[f])(t) + X(t)D^+\alpha$, $t \in [a, b]$, $\forall\, c_r \in R^r$, *where* $X_r(t)$ *is the* $(n\times n)$*-matrix whose columns consist of a full system of* $r$ *linearly independent solutions of the homogeneous system of second order* (3)*:* $X_r(t) = X(t)P_{D_r}$*;* $P_{D_r}$ *is the* $(2n\times r)$*-dimensional matrix-orthoprojector consisting of* $r$ *linearly independent columns of the matrix* $P_D$*;* $c_r$ *is an arbitrary vector column from the space* $R^r$*;* $(G[f])(t)$, $t \in [a, b]$ *is the Greens generalized operator acting onto an arbitrary vector-function* $f(t) \in C([a, b])$*:*

$$(G[f])(t) \overset{def}{=} \int_a^b K(t, s)P^{-1}(s)f(s)\,ds - X(t)D^+ l \int_a^b K(\,\cdot\,, s)P^{-1}(s)f(s)\,ds.$$

We have to define whether there exist the conditions under fulfillment of which the boundary value problem with perturbation (1), (2) will be solvable under the condition that its generating boundary value problem (3), (4) has no solutions. We consider the case, where the generating boundary value problem (3), (4) has no solutions for arbitrary inhomogeneities $f(t) \in C([a, b])$ and $\alpha \in R^m$; this implies that for the above problem the critical case($\operatorname{rank} D = n_1 < n$) is valid, and respectively, for arbitrary inhomogeneities $f(t) \in C([a, b])$, $\alpha \in R^m$, for the generating boundary value problem (3), (4) the solvability criterion (5) fails to be fulfilled. For the boundary value problem (1), (2) using the $(d\times r)$-measurable matrix $B_0 := P_{D_d^*}\big\{l_1X_r(\,\cdot\,) - l\int_a^b K(\,\cdot\,, s)P^{-1}(s)Q_1(s)X_r(s)\,ds\big\}$, the conditions of solvability of the problem under consideration and the condition of uniqueness of its solution, having the form of converging Laurent series $x(\,\cdot\,, \varepsilon) = \sum_{k=-1}^{\infty} \varepsilon^k x_k(t)$, are found. Here, $P_{B_0}$ is the $(r \times r)$-dimensional matrix-orthoprojector, $P_{B_0}$: $R^r \to N(B_0)$; $B_0^*$ is the $(r \times d)$-dimensional matrix, transposed to the matrix $B_0$, $P_{B_0^*}$ is the $(d \times d)$-dimensional matrix-orthoprojector, $P_{B_0^*}$: $R^d \to N(B_0^*)$; $B_0^+$ is the $(r \times d)$-dimensional matrix, pseudo-inverse due to Moore–Penrose to the matrix $B_0$ [6]. In the case, where the condition $P_{B_0^*} = 0$ is not fulfilled, for determination of conditions of solvability of the problem under consideration, the $(d \times r)$-measurable matrix $B_1$: $B_1 := P_{D_d^*}\big\{l_1G_1(\,\cdot\,) - l\int_a^b K(\,\cdot\,, s)P^{-1}(s)Q_1(s)G_1(s)\,ds\big\}$ has been constructed, where $G_1(t)$ is the $(n \times r)$-dimensional matrix of the type $G_1(t) = (G[Q_1(s)X_r(s)])(t) + X(t)D^+l_1X_r(\,\cdot\,)$. Here, $B_1^*$ is the $(r \times d)$-dimensional matrix, transposed to the matrix $B_1$; $P_{B_1^*}$ is the $(d \times d)$-dimensional matrix-orthoprojector, $P_{B_1^*}$: $R^d \to N(B_1^*)$. In the case, where the conditions $P_{B_0^*} = 0$,

$P_{B_1^*}P_{B_0^*}=0$ for the problem (1), (2) are not fulfilled, to find the conditions of solvability of that problem, the $(d\times r)$-dimensional matrix $\overline{B_1}:=-P_{B_0^*}B_1P_{B_0}$ has been constructed. The following theorem is valid.

**Theorem 2.** *Let the generating boundary value problem* (3), (4) *for arbitrary inhomogeneities* $f(t)\in C([a,b])$ *and* $\alpha\in R^m$ *have no solutions. For the boundary value problem* (1), (2) *the conditions* $P_{B_0^*}\neq 0$, $P_{B_1^*}P_{B_0^*}\neq 0$*) are fulfilled.*

*Then the boundary value problem with perturbation* (1), (2) *is solvable if the condition* $P_{\overline{B}_1^*}P_{B_0^*}=0$ *is fulfilled, and in this case, for a sufficiently small fixed* $\varepsilon\in(0,\varepsilon_0]$ *it has a solution in a form of a part of converging Laurent's series* $x(\,\cdot\,,\varepsilon)=\sum\limits_{k=-3}^{\infty}\varepsilon^k x_k(t)$, *the coefficients* $x_k$, $k\geq -3$ *of Laurent's series are sought from the corresponding boundary value problems constructed after substitution of the Laurent's series into the problem* (1), (2) *and equating the corresponding coefficients for each from powers* $\varepsilon$.

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