On the Existence of a Special Type Integral Manifold of a Quasilinear Differential System

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Let

$$G(\varepsilon_0)=\big\{t,\varepsilon:\ t\in\mathbf{R},\ \varepsilon\in[0,\varepsilon_0],\ \varepsilon_0\in\mathbf{R}^+\big\}.$$

Definition 1. We say that a function $f(t,\varepsilon)$, in general a complex-valued, belongs to the class $S_m(\varepsilon_0)$, $m\in\mathbf{N}\cup\{0\}$ if:

1) $f:G(\varepsilon_0)\to\mathbf{C}$;

2) $f(t,\varepsilon)\in C^m(G(\varepsilon_0))$ with respect to t;

3) $d^kf(t,\varepsilon)/dt^k=\varepsilon^kf_k^*(t,\varepsilon)$ $(0\le k\le m)$,

$$\|f\|_m\overset{def}{=}\sum_{k=0}^m\sup_{G(\varepsilon_0)}|f_k^*(t,\varepsilon)|<+\infty.$$

Definition 2. We say that a function $f(t,\varepsilon,\theta)$ belongs to the class $F^\theta_{m,\infty}(\varepsilon_0)$ $(m\in\mathbf{N}\cup\{0\})$ if this function can be represented as

$$f(t,\varepsilon,\theta)=\sum_{n=-\infty}^{\infty}f_n(t,\varepsilon)e^{in\theta},$$

and

1) $f_n(t,\varepsilon)\in S_m(\varepsilon_0)$, $\theta\in\mathbf{R}$;

2)
$$\|f_0\|_m+\sum_{\substack{n=-\infty\\(n\neq0)}}^{\infty}|n|^l\cdot\|f_n\|_m<+\infty\ \ (l=0,1,2,\dots).$$

If the function $f(t,\varepsilon,\theta)$ is real, then $f_{-n}(t,\varepsilon)=\overline{f_n(t,\varepsilon)}$.
We denote

$$\|f\|_{m,\theta}=\sum_{n=-\infty}^{\infty}\|f_n\|_m.$$

For any $f\in F^\theta_{m,\infty}(\varepsilon_0)$, we introduce the linear operators:

$$\Gamma_n[f(t,\varepsilon,\theta)]=\frac{1}{2\pi}\int\limits_0^{2\pi}f(t,\varepsilon,\theta)e^{-in\theta}\,d\theta,\ \ n\in\mathbf{Z},$$

in particular,

$$\Gamma_0[f(t,\varepsilon,\theta)] = \frac{1}{2\pi}\int\limits_0^{2\pi} f(t,\varepsilon,\theta)\,d\theta,$$

$$I[f(t,\varepsilon,\theta)] = \sum_{\substack{n=-\infty \\ (n\neq 0)}}^{\infty} \frac{\Gamma_n[f(t,\varepsilon,\theta)]}{in}\, e^{in\theta}.$$

We note some properties of the functions of the class $F^\theta_{m,\infty}(\varepsilon_0)$. Let $u(t,\varepsilon,\theta)$, $v(t,\varepsilon,\theta)$ belongs to the class $F^\theta_{m,\infty}(\varepsilon_0)$, and $k = const$.

1) $u(t,\varepsilon,\theta)$ is 2π-periodic with respect to θ.

2)
$$\frac{\partial^l u(t,\varepsilon,\theta)}{\partial\theta^l} \in F^\theta_{m,\infty}(\varepsilon_0) \ \ (l = 0,1,2,\dots),$$
$$\frac{\partial^k u(t,\varepsilon,\theta)}{\partial t^k} \in F^\theta_{k-1,\infty}(\varepsilon_0) \ \ (k = 1,\dots,m).$$

3) $\Gamma_n[u(t,\varepsilon,\theta)] \in S_m(\varepsilon_0)$ $(n \in \mathbf{Z})$.

4)
$$I[u(t,\varepsilon,\theta)] \in F^\theta_{m,\infty}(\varepsilon_0), \quad I\Big[\frac{\partial u(t,\varepsilon,\theta)}{\partial\theta}\Big] = u(t,\varepsilon,\theta) - \Gamma_0[u(t,\varepsilon,\theta)] \in F^\theta_{m,\infty}(\varepsilon_0).$$

5) $\|ku\|_{m,\theta} = |k|\cdot\|u\|_{m,\theta}$.

6) $\|u+v\|_{m,\theta} \le \|u\|_{m,\theta} + \|v\|_{m,\theta}$.

7)
$$\|u\|_{m,\theta} = \sum_{k=0}^{m} \Big\|\frac{1}{\varepsilon^k}\,\frac{\partial^k u}{\partial t^k}\Big\|_{0,\theta}.$$

8)
$$\|uv\|_{m,\theta} \le 2^m\|u\|_{m,\theta}\cdot\|v\|_{m,\theta}.$$

9) If u, v are real, then $u(t,\varepsilon,\theta + v(t,\varepsilon,\theta)) \in F^\theta_{m,\infty}(\varepsilon_0)$.

10) The chains of includes are true:
$$F^\theta_{0,\infty}(\varepsilon_0) \supset F^\theta_{1,\infty}(\varepsilon_0) \supset \cdots \supset F^\theta_{m,\infty}(\varepsilon_0), \quad S_0(\varepsilon_0) \supset S_1(\varepsilon_0) \supset \cdots \supset S_m(\varepsilon_0).$$

Definition 3. We say that a vector $f = \operatorname{colon}(f_1,\dots,f_N)$ belongs to the class $F^\theta_{m,\infty}(\varepsilon_0)$ (or $S_m(\varepsilon_0)$) if $f_j \in F^\theta_{m,\infty}(\varepsilon_0)$ (relatively, $f_j \in S_m(\varepsilon_0)$) $(j = 1,\dots,N)$.

Definition 4. We say that a matrix $(a_{jk})_{j,k=\overline{1,N}}$ belongs to the class $F^\theta_{m,\infty}(\varepsilon_0)$ (or $S_m(\varepsilon_0)$) if $a_{j,k} \in F^\theta_{m,\infty}(\varepsilon_0)$ (relatively, $a_{j,k} \in S_m(\varepsilon_0)$) $(j,k = 1,\dots,N)$.

Consider the system of differential equations:

$$\begin{aligned} \frac{dx}{dt} &= (\Lambda(t,\varepsilon) + \mu P(t,\varepsilon,\theta))x + f(t,\varepsilon,\theta), \\ \frac{d\theta}{dt} &= \omega(t,\varepsilon) + \mu a(t,\varepsilon,\theta), \end{aligned} \tag{1}$$

where $(t,\varepsilon) \in G(\varepsilon_0)$, $x = \operatorname{colon}(x_1,\dots,x_N)$, $\Lambda = \operatorname{diag}(\lambda_1,\dots,\lambda_N) \in S_m(\varepsilon_0)$, $P = (p_{jk})_{j,k=\overline{1,N}} \in F^{\theta}_{m,\infty}(\varepsilon_0)$, scalar real functions $\omega \in S_m(\varepsilon_0)$, $\inf\limits_{G(\varepsilon_0)} \omega > 0$, $a \in F^{\theta}_{m,\infty}(\varepsilon_0)$, $\mu \in (0,\mu_0) \subset \mathbf{R}^+$.

We study the problem of the conditions of existence of integral manifold $x(t,\varepsilon,\theta,\mu)$ of the system (1), belongs to the class $F^{\theta}_{m^*,\infty}(\varepsilon_0)$, where $m^* \le m$.

Lemma 1. *There exists $\mu_1 \in (0,\mu_0)$ such that for all $\mu \in (0,\mu_1)$ there exists the real reversible transformation*

$$\theta = \varphi + \mu v(t,\varepsilon,\varphi,\mu), \tag{2}$$

where $v \in F^{\varphi}_{m,\infty}(\varepsilon_0)$, reducing the system (1) to the kind

$$\begin{aligned} \frac{dx}{dt} &= (\Lambda(t,\varepsilon) + \mu Q(t,\varepsilon,\varphi,\mu))x + g(t,\varepsilon,\varphi,\mu), \\ \frac{d\varphi}{dt} &= \omega(t,\varepsilon) + \mu b(t,\varepsilon,\mu) + \mu\varepsilon\beta(t,\varepsilon,\varphi,\mu), \end{aligned} \tag{3}$$

where $Q = P(t,\varepsilon,\varphi + \mu v(t,\varepsilon,\varphi,\mu)) \in F^{\varphi}_{m,\infty}(\varepsilon_0)$, $g = f(t,\varepsilon,\varphi + \mu v(t,\varepsilon,\varphi,\mu)) \in F^{\varphi}_{m,\infty}(\varepsilon_0)$, $b \in S_m(\varepsilon_0)$, $\beta \in F^{\varphi}_{m-1,\infty}(\varepsilon_0)$.

Lemma 2. *There exists $\mu_2 \in (0,\mu_1)$ such that for all $\mu \in (0,\mu_2)$ there exists the chain of reversible transformations of kind*

$$\varphi = \psi_1 + \mu\varepsilon w_1(t,\varepsilon,\psi_1,\mu), \tag{4}$$

$$\psi_1 = \psi_2 + \mu\varepsilon^2 w_2(t,\varepsilon,\psi_2,\mu), \tag{5}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\psi_{m_1-2} = \psi_{m_1-1} + \mu\varepsilon^{m_1-1} w_{m_1-1}(t,\varepsilon,\psi_{m_1-1},\mu), \tag{6}$$

where $m_1 < m$, $w_k \in F^{\psi_k}_{m-k,\infty}(\varepsilon_0)$ $(k = 1,\dots,m_1-1)$, reducing the system (3) to the kind:

$$\begin{aligned} \frac{dx}{dt} &= \big(\Lambda(t,\varepsilon) + \mu R_{m_1-1}(t,\varepsilon,\psi_{m_1-1},\mu)\big)x + h_{m_1-1}(t,\varepsilon,\psi_{m_1-1},\mu), \\ \frac{d\psi_{m_1-1}}{dt} &= \omega(t,\varepsilon) + \mu b(t,\varepsilon,\mu) + \mu \sum_{l=1}^{m_1-1} \varepsilon^k \beta_k(t,\varepsilon,\mu) + \mu\varepsilon^{m_1}\widetilde{\beta}_{m_1-1}(t,\varepsilon,\psi_{m_1-1},\mu), \end{aligned} \tag{7}$$

where $R_{m_1-1} \in F^{\psi_{m_1-1}}_{m-m_1+1}(\varepsilon_0)$, $h_{m_1-1} \in F^{\psi_{m_1-1}}_{m-m_1+1}(\varepsilon_0)$, $\beta_k \in S_{m-k}(\varepsilon_0)$, $\widetilde{\beta}_{m_1-1} \in F^{\psi_{m_1-1}}_{m-m_1,\infty}(\varepsilon_0)$ $(k = 1,\dots,m_1-1)$.

Theorem. *Let the elements $\lambda_j(t,\varepsilon)$ $(j = 1,\dots,N)$ of matrix $\Lambda(t,\varepsilon)$ in system (1) be such that*

$$\inf_{G(\varepsilon_0)} |\operatorname{Re}\lambda_j(t,\varepsilon)| \ge \gamma > 0 \ \ (j = 1,\dots,N).$$

Then there exists $\mu^* \in (0,\mu_0)$ such that for all $\mu \in (0,\mu^*)$ the system (1) has the integral manifold $\widetilde{x}(t,\varepsilon,\theta,\mu) \in F^{\theta}_{m_1,\infty}(\varepsilon_0)$, where $2m_1 \le m$ $(m_1 \in \mathbf{N} \cup \{0\})$.

Proof. Based on Lemmas 1, 2, we reduce the system (1) to the kind (7). We denote

$$\psi = \psi_{m_1-1},$$
$$R(t,\varepsilon,\psi,\mu) = R_{m_1-1}(t,\varepsilon,\psi_{m_1-1},\mu), \quad h(t,\varepsilon,\psi,\mu) = h_{m_1-1}(t,\varepsilon,\psi_{m_1-1},\mu),$$
$$\omega_1(t,\varepsilon,\mu) = \omega(t,\varepsilon_0) + \mu b(t,\varepsilon,\mu) + \sum_{l=1}^{m_1-1} \varepsilon_k\beta_k(t,\varepsilon,\mu) + \varepsilon^{m_1}\widetilde{\beta}_{m_1-1}(t,\varepsilon,\psi_{m_1-1},\mu).$$

Based on condition of Theorem and property 10) of the functions of class $F^{\theta}_{m,\infty}(\varepsilon_0)$, we can state that $R(t,\varepsilon,\psi,\mu)$, $h(t,\varepsilon,\psi,\mu) \in F^{\psi}_{m_1,\infty}(\varepsilon_0)$, $\omega_1(t,\varepsilon,\mu) \in S_{m_1}(\varepsilon_0)$. Then we write the system (7) in kind

$$\begin{aligned} \frac{dx}{dt} &= (\Lambda(t,\varepsilon) + \mu R(t,\varepsilon,\psi,\mu))x + h(t,\varepsilon,\psi,\mu), \\ \frac{d\psi}{dt} &= \omega_1(t,\varepsilon,\mu). \end{aligned} \tag{8}$$

With the system (8) consider the system

$$\begin{aligned} \frac{dx_0}{dt} &= \Lambda(t,\varepsilon)x_0 + h(t,\varepsilon,\psi,\mu), \\ \frac{d\psi}{dt} &= \omega_1(t,\varepsilon,\mu). \end{aligned} \tag{9}$$

Based on the results [1] and condition of Theorem, we can state that the system (9) has the integral manifold $x_0(t,\varepsilon,\psi,\mu) \in F^{\psi}_{m_1,\infty}(\varepsilon_0)$. And there exists $K \in (0,+\infty)$ such that

$$\|x_0\|_{m_1,\psi} \le K\|h\|_{m_1,\psi}. \tag{10}$$

We seek the integral manifold of system (8) by the method of succesive approximations, defining as an initial approximation x_0, and the subsequents approximations defining from the systems:

$$\begin{aligned} \frac{dx_{s+1}}{dt} &= \Lambda(t,\varepsilon)x_{s+1} + h(t,\varepsilon,\psi,\mu) + \mu R(t,\varepsilon,\psi,\mu)x_s, \\ \frac{d\psi}{dt} &= \omega_1(t,\varepsilon,\mu), \quad s = 0,1,2,\dots. \end{aligned} \tag{11}$$

Based on inequality (10) and using the ordinary technicue of the contraction mapping principle [2], it is easy to show that there exists $\mu_3 \in (0,\mu_0)$ such that for all $\mu \in (0,\mu_3)$ all approximations x_s belong to the class $F^{\psi}_{m_1,\infty}(\varepsilon_0)$, and process (11) converges by the norm $\|\cdot\|_{m,\psi}$ to integral manifold $x(t,\varepsilon,\psi,\mu) \in F^{\psi}_{m_1,\infty}(\varepsilon_0)$ of the system (8).

Based on the reversibility of the transformations (2), (4)–(6), we can state the existence of the integral manifold $\widetilde{x}(t,\varepsilon,\theta,\mu) \in F^{\theta}_{m,\infty}(\varepsilon_0)$ of the system (1) for sufficiently small values μ. □

References

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[2] A. M. Kolmogorov and S. V. Fomin, Elements of the theory of functions and functional analysis (Russian) *Nauka, Moscow*, 1972.