Global Attractors for Some Class of Discontinuous Dynamical Systems

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An autonomous evolution system is called discontinuous (or impulsive) dynamical system (DS) if its trajectories have jumps at moments of intersection with certain surface of the phase space [8]. Some aspects of qualitative behavior of impulsive finite-dimensional DS have been studied in recent years [8, 5, 4]. For infinite-dimensional dissipative DS one of the most important problems is investigation of global attractor [9]. This approach has been applied to impulsive DS in [1, 2, 7]. In this paper we propose a new concept of global attractor for discontinuous DS, which is based on the definition of uniform attractor for non-autonomous DS [3], in particular, for systems with impulsive perturbation at fixed moment of time [6]. Using this concept, we investigate asymptotic behavior of a wide class of dissipative infinite-dimensional impulsive DS, generated by parabolic equation with impulsive perturbations at non-fixed moments of time. We consider existence and non-existence results in the case of linear equation and we also give effective sufficient conditions for existence of global attractor in the case of weakly nonlinear parabolic equation.

Let (X,ρ) be a metric space, $P(X)$ ($\beta(X)$) be a set of all nonempty (nonempty bounded) subsets of X,

$$\operatorname{dist}_X(A,B) := \sup_{y\in A}\inf_{z\in B}\rho(y,z).$$

A pair (X,G) is called dynamical system (DS) if

$$\forall\, x\in X \ \ G(0,x)=x, \quad G(t+s,x)=G(t,G(s,x)) \ \ \forall\, t,s\geq 0.$$

We assume no conditions of continuity for the map $x\to G(t,x)$.

Definition. A set $\Theta\subset X$ is called global attractor of DS (X,G) if

1) Θ is a compact set;
2) Θ is uniformly attracting set, i.e.,

$$\forall\, B\in\beta(X) \ \ \operatorname{dist}_X(G(t,B),\Theta)\to 0, \ \ t\to\infty;$$

3) Θ is minimal among closed sets satisfying 2).

Theorem 1. *Suppose DS (X,G) satisfies dissipativity condition:*

$$\exists\, B_0\in\beta(X) \ \ \forall\, B\in\beta(X) \ \ \exists\, T=T(B) \ \ \forall\, t\geq T \ \ G(t,B)\subset B_0.$$

Then DS (X,G) has global attractor if and only if

$$\forall\,\{x_n\}\in\beta(X) \ \ \forall\,\{t_n\nearrow\infty\} \ \ \textit{sequence} \ \ \{G(t_n,x_n)\} \ \ \textit{is precompact}.$$

Now we construct DS, generated by impulsive system. It is called impulsive DS and consists of classical (continuous) DS (X,V), a closed set $M\subset X$ (impulsive set) and a map $I:M\to X$ (impulsive map). The phase point $x(t)$ moves along trajectories of DS (X,V) and when it reaches the set M at the moment τ, it jumps to a new position $Ix(\tau)$.

We assume the following conditions hold:

$$M \cap I(M) = \varnothing, \ \ \forall\, x \in M \ \ \exists\, \tau = \tau(x) > 0 \ \ \forall\, t \in (0,\tau) \ \ V(t,x) \notin M.$$

We define $\forall\, x \in M \ \ Ix = x^+$, $\forall\, x \in X \ \ M^+(x) = \big(\bigcup\limits_{t>0} V(t,x)\big) \cap M$.

It follows from continuity of V that if $M^+(x) \neq \varnothing$, then there exists $s := \phi(x) > 0$ such that

$$\forall\, t \in (0,s) \ \ V(t,x) \notin M, \ \ V(s,x) \in M. \tag{1}$$

So, for fixed $x \in X$ we have:

- if $M^+(x) = \varnothing$, then $\widetilde{V}(t,x) = V(t,x) \ \ \forall\, t \geq 0$;
- if $M^+(x) \neq \varnothing$, then for $s_0 = \phi(x)$, $x_1 = V(s_0,x)$

$$\widetilde{V}(t,x) = \begin{cases} V(t,x), & 0 \leq t < s_0, \\ x_1^+, & t = s_0; \end{cases}$$

- if $M^+(x_1^+) = \varnothing$, then $\widetilde{V}(t,x) = V(t-s_0, x_1^+) \ \forall\, t \geq s_0$;
- if $M^+(x_1^+) \neq \varnothing$, then for $s_1 = \phi(x_1^+)$, $x_2 = V(s_1, x_1^+)$

$$\widetilde{V}(t,x) = \begin{cases} V(t-s_0, x_1^+), & s_0 \leq t < s_0 + s_1, \\ x_2^+, & t = s_0 + s_1, \end{cases}$$

and so on. As a result, we obtain finite or infinite number of impulsive points $\{x_n^+\}_{n\geq 1}$ and corresponding moments of time $\{s_n\}_{n\geq 0}$ such that

$$V(s_0,x) = x_1, \quad V(s_n, x_n^+) = x_{n+1}, \ \ n \geq 1.$$

Let us assume the following condition holds:

$$\forall\, x \in X \ \ \widetilde{V}(t,x) \ \text{ is well-defined on } \ [0,+\infty).$$

This condition means that either the number of impulsive points is finite or $\sum\limits_{n=0}^{\infty} s_n = \infty$. Then [5] the map $\widetilde{V} : R_+ \times X \mapsto X$ satisfies semigroup property and we have impulsive DS $(X, \widetilde{V})$.

We study global attractors of impulsive DS $(X,\widetilde{V})$ in the following two cases [8]:

(a) X is a Banach space, $M = \{x \in X \mid \ \|x\| = a\}$, $Ix = (1+\mu)x$, where $a > 0$, $\mu > 0$;

(b) X is a Hilbert space, $\{\psi_k\}_{k=1}^{\infty}$ is an orthonormal basis in H, $M = \{x \in X \mid \ (\psi_1, x) = a\}$, and for $x = \sum\limits_{k=1}^{\infty} c_k\psi_k$, $Ix = (1+\mu)c_1\psi_1 + \sum_{k=2}^{\infty} c_k\psi_k$.

At first we illustrate some interesting properties in linear case: in bounded domain $\Omega \subset R^p$, $p \geq 1$ we consider the linear problem

$$\begin{cases} \dfrac{\partial y}{\partial t} = \Delta y, \ \ (t,x) \in (0,\infty) \times \Omega, \\ y\big|_{\partial\Omega} = 0. \end{cases} \tag{2}$$

Let $\{\psi_i\}_{i=1}^{\infty}$ be a complete and orthonormal in $L^2(\Omega)$ family of eigenfunctions of $-\Delta$, i.e., $-\Delta\psi_i = \lambda_i\psi_i$, $\psi_i \in H_0^1(\Omega)$, $0 < \lambda_1 \leq \lambda_2 \leq \cdots$, $\lambda_i \to \infty$, $i \to \infty$.

The problem (2) in the phase space $X = L^2(\Omega)$ with a norm $\|\cdot\|$ and a scalar product $(\cdot,\cdot)$ generates classical DS (X,V), where

$$y(t) = V\Big(t, \sum_{i=1}^{\infty} c_i\psi_i\Big) = \sum_{i=1}^{\infty} c_i e^{-\lambda_i t}\psi_i.$$

As $\forall\, t \ge 0$ $\|y(t)\| \le e^{-\lambda_1 t}\|y_0\|$, then DS (X,V) has a trivial global attractor $\Theta = \{0\}$.

Now let us consider impulsive DS $(X,\widetilde{V})$, where

$$M = \big\{y \in X \mid \|y\| = \varepsilon\big\}, \;\; Iy = (1+\mu)y, \;\; \varepsilon > 0, \;\; \mu > 0. \tag{3}$$

Lemma 1. *For every $\varepsilon > 0$, $\mu > 0$ the problem* (2), (3) *generates dissipative impulsive DS $(X,\widetilde{V})$, which does not possess global attractor.*

Let us consider impulsive DS $(X,\widetilde{V})$, where

$$M = \{y \in X \mid (y,\psi_1) = a\}, \;\; I : M \mapsto X, \tag{4}$$

$$\text{for } y = \sum_{i=1}^{\infty} c_i\psi_i, \;\; Iy = (\mu+1)c_1\psi_1 + \sum_{i=2}^{\infty} c_i\psi_i, \;\; a > 0, \;\; \mu > 0.$$

Lemma 2. *For every $a > 0$, $\mu > 0$ the problem* (2), (4) *generates dissipative impulsive DS $(X,\widetilde{V})$, which has global attractor*

$$\Theta = \bigcup_{t\in[0,\ln(1+\mu)]} \big\{(1+\mu)ae^{-t}\psi_1\big\} \cup \{0\}. \tag{5}$$

From (5) we can see that $\Theta \cap M \ne \varnothing$ and $\forall\, t > 0$ $\widetilde{V}(t,\Theta) \not\subset \Theta$. But invariance property is true in the following form:

$$\forall\, t > 0 \;\; \widetilde{V}(t,\Theta \setminus M) \subset \Theta \setminus M. \tag{6}$$

The main result of the work is to prove that the statements of Lemma 2 remain true in nonlinear case.

In bounded domain $\Omega \subset R^p$, $p \ge 1$ we consider the problem

$$\begin{cases} \dfrac{\partial y}{\partial t} = \Delta y - \varepsilon f(y), \;\; (t,x) \in (0,\infty)\times\Omega, \\ y\big|_{\partial\Omega} = 0, \end{cases} \tag{7}$$

where $\varepsilon > 0$ is a small parameter, $f \in C^1(R)$, $f(0) = 0$,

$$\exists\, C > 0 \;\; \forall\, y \in R, \;\; f'(y) \ge -C, \;\; |f(y)| \le C. \tag{8}$$

Under conditions (8) for arbitrary $y_0 \in X = L^2(\Omega)$ the problem (7) has a unique solution $y_\varepsilon \in C([0,+\infty);X)$, $y_\varepsilon(0) = y_0$.

Theorem 2. *For every $a > 0$, $\mu > 0$ and for sufficiently small $\varepsilon > 0$ impulsive problem* (7), (4) *generates impulsive DS $(X,\widetilde{V}_\varepsilon)$, which has global attractor Θ_ε and, moreover,*

$$\operatorname{dist}(\Theta_\varepsilon,\Theta) \to 0, \;\; \varepsilon \to 0, \tag{9}$$

$$\forall\, t > 0 \;\; \widetilde{V}_\varepsilon(t,\Theta_\varepsilon \setminus M) \subset \Theta_\varepsilon \setminus M. \tag{10}$$

Proof. For every solution of (7) $y_\varepsilon(\cdot)$ we have

$$\forall\, t \ge 0 \;\; (y_\varepsilon(t),\psi_1) = e^{-\lambda_1 t}(y_\varepsilon(0),\psi_1) - \varepsilon\int_0^t e^{-\lambda_1(t-p)}(f(y_\varepsilon(p)),\psi_1)\,dp. \tag{11}$$

Equality (11) allows us to estimate the moments of impulsive perturbation of every trajectory of (7), (4) with the help of Implicit Function Theorem. Then we prove existence of global attractor and limit equality (9). To prove invariance property (10) we consider function $x \mapsto \phi(x)$, defined in (1), and we show its continuity on $X \setminus M$. $\square$

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