Stability of Trivial Invariant Torus of Dynamical System

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1 Introduction and Preliminaries

We consider an autonomous system of differential equations

$$\dot{x} = F(x), \quad x \in \mathbb{R}^k \tag{1}$$

that possesses m-dimensional invariant toroidal manifold $\mathcal{T}_m$. For a comprehensive description of the dynamics in the vicinity of invariant toroidal manifold it is convenient to introduce so-called local coordinates $(\varphi_1, \dots, \varphi_m, h_1, \dots, h_n)$, $n = k - m$, where $\varphi = (\varphi_1, \dots, \varphi_m)$ is a point on torus $\mathcal{T}_m$ and $h = (h_1, \dots, h_n)$ is from Euclidean space in transversal direction to the torus. The change of variables is performed in such a way that the invariant toroidal manifold gets a representation $h = 0, \varphi \in \mathcal{T}_m$ in new coordinates. System (1) transforms into

$$\dot{\varphi} = a(\varphi, h), \quad \dot{h} = f(\varphi, h) \tag{2}$$

with $f(\varphi, 0) \equiv 0$. The last condition guarantees the existence of invariant toroidal set $h = 0, \varphi \in \mathcal{T}_m$ that is called trivial.

Problems of the existence, stability and an approximate construction of non-trivial invariant toroidal manifolds for system (2) are treated carefully in [10]. The central object of investigation is a so-called linear extension of dynamical system on torus

$$\dot{\varphi} = a(\varphi), \quad \dot{h} = A(\varphi)h + f(\varphi), \tag{3}$$

where $a \in C_{Lip}(\mathcal{T}_m)$ is an m-dimensional vector function, $A, f \in C(\mathcal{T}_m)$ are $n \times n$ square matrix and n-dimensional vector function respectively; $C(\mathcal{T}_m)$ stands for a space of continuous 2π-periodic with respect to each of the variables φ_j, $j = 1, \dots, m$ functions defined on the surface of the torus $\mathcal{T}_m$. The main ingredient in the investigation of the existence and stability analysis of non-trivial invariant tori of system (3) is Green function introduced in [8]. The existence of such a function is sufficient for the existence of non-trivial invariant torus for system (3). Later a numerous of works by different authors have developed and extended this approach to a broad classes of equations including impulsive [4, 3], stochastic [11] and infinite-dimensional [7] and equations with delay [9]. This method of investigation got a *Green-Samoilenko function method* name [7].

A deep connection of the existence of invariant tori and quadratic functions was explored in [1]. A Lyapunov-like approach was proposed for stability analysis of invariant tori and their robustness properties characterization. A question of the preservation of invariant tori under perturbations of the right-hand side was also considered. It has been proven that a sufficiently small perturbations do not ruin the invariant torus, which enables it to become a convenient object for investigations of quasi-periodic motions of dynamical system. As it is widely known, quasi-periodic solution may be easily transformed into a periodic one by a small perturbation of right-hand side. The existence of invariant tori that is a carrier of quasi-periodic trajectories ensures the existence of multi-frequency oscillations in the system. It makes this theory well-adapted for the applications in electronics and radiophysics with complex oscillatory processes of several frequencies.

2 Motivation

In this paper we are interested in stability analysis of trivial invariant torus of the system

$$\dot{\varphi} = a(\varphi), \quad \dot{h} = A(\varphi, h)h, \tag{4}$$

where $\varphi \in \mathcal{T}_m$, $h \in \mathbb{R}^n$.

We begin with a simple example that demonstrates that the existing theorems for stability analysis of invariant tori are too restrictive and set too severe constraints on the system. On the other hand, we propose relaxed conditions that are applicable to a wide class of equations and provide a deeper understanding of the processes in a vicinity of invariant set.

Example 1. Consider system (4) with $a(\varphi) = \begin{pmatrix} -\sin^2 \frac{\varphi_1}{2} \\ \omega \end{pmatrix}$ and $A(\varphi, h) = -1 + \lambda \sin \varphi_1$, where $\lambda > 0$ is an arbitrary fixed constant value from $\mathbb{R}$.

System from the example may be analyzed in two steps:

$$\dot{\varphi} = a(\varphi), \quad \dot{h} = -1 \cdot h \implies \dot{\varphi} = a(\varphi), \quad \dot{h} = (-1 + \lambda \sin \varphi_1)h, \tag{5}$$

where $\lambda \cos \varphi$ is considered as a perturbation term. The fundamental matrix $\Omega^t_\tau(\varphi)$ of the system $\dot{h} = -h$ has a form $\Omega^t_\tau(\varphi) = e^{-(t-\tau)}$. It means that system $\dot{\varphi} = a(\varphi)$, $\dot{h} = -h$ has an exponentially stable trivial invariant torus $h = 0$, $\varphi \in \mathcal{T}_m$. The previously known perturbation theorems guarantee stability of trivial torus of system (5) in the case of a sufficiently small perturbation term, e.g. there exists $\delta > 0$ such that for any perturbation with $\|\lambda \cos \phi\| \leq \delta$ system (5) has an exponentially stable trivial invariant toroidal manifold. In other words, a stability of manifold is guaranteed only for a sufficiently small constant λ. However a numerical simulations provides an intuition that the trivial torus is actually asymptotically stable even for large enough values of the constant parameter λ. Indeed, for the cases of $\lambda = 1$, $\lambda = 10$, and $\lambda = 100$ a qualitative behavior of solutions to system (5) coincide and all the trajectories tend to the invariant set as time $t \to \infty$. This fact originate a hypothesis that a smallness of a perturbation term is too severe constraint and can be relaxed. This is the main motivation for this research.

Further propositions deeply rely on the results from [5, 6].

3 Results

Denoting $A(\varphi, h) := A(\varphi) + A_1(\varphi)$, system (4) can be represented in the following form

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dh}{dt} = [A(\varphi) + A_1(\varphi, h)]h, \tag{6}$$

where A_1 is a perturbation term from $C(\mathcal{T}_m, \mathbb{R}^n)$, $\|h\| \leq d \in \mathbb{R}_+$. Let $\mathcal{H}^t_\tau(\varphi)$ be a fundamental matrix of the unperturbed system

$$\frac{d\varphi}{dt} = a(\varphi), \quad \frac{dh}{dt} = A(\varphi)h,$$

that depends on $\varphi \in \mathcal{T}_m$ as a parameter and turns into an identical matrix when $t = \tau$, e.g. $\mathcal{H}^\tau_\tau(\varphi) \equiv I$.

Definition 1 ([2])**.** A point φ is called wandering if there exist its neighbourhood $U(\varphi)$ and a positive number $T > 0$ such that

$$U(\varphi) \cap \varphi_t(U(\varphi)) = 0 \ \text{ for } \ t \geq T.$$

Let W be a set of all wandering points of dynamical system and $\Omega = \mathcal{T}_m \setminus W$ be a set of nonwandering points. From the compactness of a torus it follows that the set Ω is nonempty and compact. Since function $A_1(\varphi, h)$ is continuous on a compact set, there exists

$$\sup_{\varphi\in\Omega, \|h\|\leq d} A_1(\varphi, h) = \widetilde{a}_1.$$

The following proposition sets constraints on the perturbation term in order to guarantee the exponential stability of the trivial invariant torus $h = 0, \varphi \in \mathcal{T}_m$. These constraints are relaxed comparing to the previously known [1, 10] and demand the perturbation to be small in non-wandering set of dynamical system Ω, but not on the whole surface of the torus $\mathcal{T}_m$.

Theorem 1. *Let the fundamental matrix $\mathcal{H}_\tau^t(\varphi)$ satisfy the estimate*

$$\| \mathcal{H}_\tau^t(\varphi)\| \leq Ke^{-\gamma(t-\tau)} \ \textit{ for } \ t \geq \tau$$

with some $K \geq 1$, $\gamma > 0$. Then if the following condition holds

$$K\widetilde{a}_1 < \gamma,$$

then system (6) has an exponentially stable trivial invariant toroidal manifold.

Example 2 (revisited)**.** The dynamical system on two-dimensional torus

$$\begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \end{pmatrix} = \begin{pmatrix} -\sin^2 \dfrac{\varphi_1}{2} \\ \omega \end{pmatrix}$$

has a very simple structure of limit sets and recurrent trajectories. In particular a non-wandering set Ω consists of only one meridian $\varphi_1 = 0$:

$$\Omega = \big\{\varphi \in \mathcal{T}_2 : \varphi_1 = 0, \varphi_2 \in \mathcal{T}_1\big\}.$$

A point that is starting on meridian spinning with constant velocity ω. All other trajectories tend to Ω by spirals. The estimate for the perturbation term is

$$\sup_{\varphi\in\Omega,\ \|x\|\leq d} \lambda \sin\varphi_1 = \lambda \sin 0 = 0.$$

It means that the system from the example and the perturbation term satisfy the conditions of Theorem 1 and the trivial invariant tori of system (4) with $a(\varphi) = \begin{pmatrix} -\sin^2 \frac{\varphi_1}{2} \\ \omega \end{pmatrix}$ and $A(\varphi, h) = -1 + \lambda \sin\varphi_1$ is exponentially stable for an arbitrary fixed constant λ.

4 Discussion

We have proved that it is sufficient for a perturbation term to be small only in a non-wandering set Ω in order to preserve an exponential stability of a trivial invariant torus of a perturbed system. New theorem allows to investigate qualitative behavior of solutions of a class of nonlinear systems that have a simple structure of limit sets and recurrent trajectories. The constraints of Theorem 1 are less restrictive than of the previously known ones. However it is worth to note that if the first equation of the unperturbed system is $\dot{\varphi} = \omega = const$, that is very frequent in applications, then its non-wandering set Ω coincides with a whole torus and Theorem 1 has no advantages compared to results from [1, 10].

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