The Nonlinear Kneser Problem for Singular in Phase Variables Two-Dimensional Differential Systems

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Let $a > 0$, $\mathbb{R}_- =]-\infty, 0]$, $\mathbb{R}_+ = [0, +\infty[$, and $\mathbb{R}_{0+} =]0, +\infty[$. On a positive semi-axis $\mathbb{R}_{0+}$, we consider the differential system

$$\frac{du_i}{dt} = f_i(t, u_1, u_2) \ \ (i = 1, 2) \tag{1}$$

with the boundary condition

$$\varphi(u_1) = c, \tag{2}$$

where c is a positive constant, $f_i : \mathbb{R}_{0+} \times \mathbb{R}^2_{0+} \to \mathbb{R}_-$ $(i = 1, 2)$ are continuous functions, and $\varphi : C([0, a]; \mathbb{R}_+) \to \mathbb{R}_+$ is a continuous nondecreasing functional.

A continuously differentiable vector function $(u_1, u_2) : \mathbb{R}_{0+} \to \mathbb{R}^2_{0+}$, satisfying system (1) in $\mathbb{R}_{0+}$, is said to be **a positive solution** of that system.

If the component u_i of a positive solution (u_1, u_2) at the point 0 has the right-hand limit

$$u_i(0+) = \lim_{t>0,\, t\to 0} u_i(t),$$

then we put $u_i(0) = u_i(0+)$.

A positive solution (u_1, u_2) of system (1) is said to be **a positive solution of problem** (1), (2) if there exists $u_1(0+)$ and equality (2) is satisfied.

A positive solution (u_1, u_2) of system (1) is said to be **a vanishing at infinity positive solution** if

$$\lim_{t\to+\infty} u_i(t) = 0 \ \ (i = 1, 2).$$

If

$$f_1(t, x, y) \equiv -y, \quad f_2(t, x, y) \equiv -f(t, x, -y),$$

then the differential system (1) is equivalent to the differential equation

$$u'' = f(t, u, u'), \tag{3}$$

and condition (2) is equivalent to the condition

$$\varphi(u) = c, \tag{4}$$

respectively. Consequently, problem (1), (2) has a positive solution if and only if problem (3), (4) has a so-called Kneser solution, i.e. a solution satisfying the inequalities

$$u(t) > 0, \quad u'(t) < 0 \ \text{ for } \ t \in \mathbb{R}_{0+}.$$

Problem (1), (2), as problem (3), (4), is said to be the nonlinear Kneser problem. These problems are investigated in detail in the case where the functions f_i $(i = 1, 2)$ and f have no singularities in phase variables (see, e.g., [1–6], and the references therein).

In [7], for the singular in a phase variable equation (3), sufficient conditions for the existence of a Kneser solution satisfying the condition (4) are established. Theorems below are generalizations of the above mentioned results for system (1).

Below everywhere it is assumed that the functions f_i $(i = 1, 2)$ on the set $\mathbb{R}_{0+} \times \mathbb{R}^2_{0+}$ admit the estimates

$$g_{10}(t) \le -x^{\lambda_1} y^{-\mu_1} f_1(t, x, y) \le g_1(t),$$
$$g_{20}(t) \le -x^{\lambda_2} y^{-\mu_2} f_2(t, x, y) \le g_2(t),$$

where λ_i and μ_i $(i = 1, 2)$ are nonnegative constants, and $g_{i0} : \mathbb{R}_{0+} \to \mathbb{R}_{0+}$, $g_i : \mathbb{R}_{0+} \to \mathbb{R}_{0+}$ $(i = 1, 2)$ are continuous functions. If $\lambda_i > 0$ for some $i \in \{1, 2\}$, then

$$\lim_{x\to 0} f_i(t, x, y) = +\infty \ \text{ for } \ t > 0, \ \ y > 0.$$

And if $\mu_2 > 0$, then

$$\lim_{y\to 0} f_2(t, x, y) = +\infty \ \text{ for } \ t > 0, \ \ x > 0.$$

Consequently, in both cases system (1) has the singularity in at least one phase variable.

We use the following notation and definitions.

$$\nu_0 = \frac{\mu_1}{1 + \mu_2}\,, \quad \nu = 1 + \lambda_1 + \lambda_2 \nu_0.$$

$C([0, a]; \mathbb{R})$ is the Banach space of continuous functions $u : [0, a] \to \mathbb{R}$ with the norm

$$\|u\|_C = \max\big\{|u(t)| : \ 0 \le t \le a\big\},$$

$C([0, a]; \mathbb{R}_+) = \big\{u \in C([0, a]; \mathbb{R}) : \ u(t) \ge 0 \text{ for } 0 \le t \le a\big\}$.

A functional $\varphi : C([0, a]; \mathbb{R}_+) \to \mathbb{R}_+$ is said to be **nondecreasing** if for any $u \in C([0, a]; \mathbb{R}_+)$ and $u_0 \in C([0, a]; \mathbb{R}_+)$ the inequality

$$\varphi(u + u_0) \ge \varphi(u)$$

holds.

Theorem 1. *If*

$$\int_t^{+\infty} g_{20}(s)\,ds < +\infty, \quad w_0(t) \equiv \int_t^{+\infty} g_{10}(s) \Big(\int_s^{+\infty} g_{20}(\tau)\,d\tau \Big)^{\nu_0} ds < +\infty \ \ \textit{for} \ \ t > 0,$$

and

$$w(t) \equiv \int_t^{+\infty} w_0^{-\frac{\lambda_2}{\nu}}(s) g_2(s)\,ds < +\infty, \quad \int_t^{+\infty} g_1(s) w^{\nu_0}(s)\,ds < +\infty \ \ \textit{for} \ \ t > 0,$$

then system (1) *has at least one vanishing at infinity positive solution.*

Corollary 1. *Let*

$$\liminf_{t\to+\infty} (t^{1-\alpha} g_{10}(t)) > 0, \quad \limsup_{t\to+\infty} (t^{1-\alpha} g_1(t)) < +\infty, \tag{5}$$
$$\liminf_{t\to+\infty} (t^{\beta} g_{20}(t)) > 0, \quad \limsup_{t\to+\infty} (t^{\beta} g_2(t)) < +\infty, \tag{6}$$

where α and β are nonnegative constants. Then for the existence of at least one vanishing at infinity positive solution of system (1) *it is necessary and sufficient that*

$$\beta > \frac{1 + \mu_2}{\mu_1}\,\alpha + 1.$$

If
$$\int\limits_t^{+\infty} g_2(s)\,ds < +\infty \;\text{ for }\; t > 0, \quad \int\limits_0^{+\infty} g_1(s)\bigg(\int\limits_s^{+\infty} g_2(\tau)\,d\tau\bigg)\,ds < +\infty, \tag{7}$$
then on the set $\mathbb{R}_+ \times \mathbb{R}_{0+}$ we put
$$v_0(t,x) = \bigg[x^\nu + \nu(1+\mu_2)^{\nu_0}\int\limits_t^{+\infty} g_{10}(s)\bigg(\int\limits_s^{+\infty} g_{20}(\tau)\,d\tau\bigg)^{\nu_0}\,ds\bigg]^{\frac{1}{\nu}},$$
$$v_1(t,x) = \bigg[x^{1+\lambda_1} + (1+\lambda_1)\int\limits_t^{+\infty} \nu^{\mu_1}(s,x)g_1(s)\,ds\bigg]^{\frac{1}{1+\lambda_1}},$$
where
$$v(t,x) = \bigg[(1+\mu_2)\int\limits_t^{+\infty} \nu_0^{-\lambda_2}(s,x)g_2(s)\,ds\bigg]^{\frac{1}{1+\mu_2}} \;\text{ for }\; t > 0, \;\; x > 0.$$

Theorem 2. *Let either*
$$\int\limits_t^{+\infty} g_{10}(s)\,ds = +\infty \;\textit{ for }\; t > 0,$$
or
$$\int\limits_t^{+\infty} g_{20}(s)\,ds < +\infty \;\textit{ for }\; t > 0, \quad \int\limits_0^{+\infty} g_{10}(s)\bigg(\int\limits_s^{+\infty} g_{20}(\tau)\,d\tau\bigg)^{\nu_0}\,ds < +\infty$$
and
$$\varphi(v_0(\,\cdot\,;0)) > c.$$
Then problem (1), (2) *has no solution.*

Theorem 3. *Let along with* (7) *the conditions*
$$\lim_{x\to+\infty} \varphi(x) = +\infty$$
and
$$\inf\big\{\varphi(v_1(\,\cdot\,;x)) :\; x > 0\big\} < c$$
be satisfied. Then problem (1), (2) *has at least one positive solution.*

Theorems 2 and 3 yield the following propositions.

Corollary 2. *Let*
$$\int\limits_{t_0}^{+\infty} g_{10}(s)\,ds = +\infty,$$
where $t_0 > 0$. *Then for the existence of at least one positive solution of problem* (1), (2) *for every sufficiently large* c, *it is necessary and sufficient that*
$$\int\limits_t^{+\infty} g_{20}(s)\,ds < +\infty \;\textit{ for }\; t > 0, \quad \int\limits_0^{+\infty} g_{10}(s)\bigg(\int\limits_s^{+\infty} g_{20}(\tau)\,d\tau\bigg)^{\nu_0}\,ds < +\infty.$$

Corollary 3. *Let conditions* (5) *and* (6) *hold, where* α *and* β *are nonnegative constants. Then for the existence of at least one positive solution of problem* (1), (2) *for every sufficiently large* c*, it is necessary and sufficient that*
$$\beta > \frac{1+\mu_2}{\mu_1}\,\alpha + 1.$$

Finally we note that the proofs of the above formulated theorems are based on the results of [8].

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