An Optimal Control Problem for a Class of Functional Differential Equations with Continuous and Discrete Times

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1 Introduction

Here we continue the study of functional differential systems that cover many kinds of dynamic models with aftereffect (integro-differential, delayed differential, differential difference, difference), see [5, 3] and references therein. First we recall the description of a class of continuous-discrete functional differential equations with linear Volterra operators and appropriate spaces where those are considered. On the basis of the representation of general solution to the system with the use of the Cauchy operator we consider an optimal control problem and propose sufficient and necessary conditions for its solvability in the terms of programming control.

2 A class of Continuous-Discrete Functional Differential Systems

Fix a segment $[0,T]\subset R$. By $L^n=L^n[0,T]$ we denote the space of summable functions $v:[0,T]\to R^n$ under the norm $\|v\|_{L^n}=\int\limits_0^T |v(s)|_n\,ds$, where $|\cdot|_n$ stands for the norm of R^n; $L_2^n=L_2^n[0,T]$ is the space of square summable functions $u:[0,T]\to R^r$ with the inner product $(u,v)=\int\limits_0^T u^{\perp}(s)v(s)\,ds$, where $\perp$ stands for transposition.

The space $AC^n=AC^n[0,T]$ is the space of absolutely continuous functions $x:[0,T]\to R^n$ with the norm

$$\|x\|_{AC^n}=\|\dot x\|_{L^n}+|x(0)|_n.$$

Let us fix a set $J=\{t_0,t_1,\dots,t_\mu\}$, $0=t_0<t_1<\dots<t_\mu=T$.

$FD^\nu(\mu)=FD^\nu\{t_0,t_1,\dots,t_\mu\}$ denotes the space of functions $z:J\to R^\nu$ under the norm

$$\|z\|_{FD^\nu(\mu)}=\sum_{i=0}^{\mu}|z(t_i)|_\nu.$$

We consider the system under control

$$\begin{aligned}\dot x&=\mathcal{T}_{11}x+\mathcal{T}_{12}z+Fu+f,\\ z&=\mathcal{T}_{21}x+\mathcal{T}_{22}z+g,\end{aligned}\tag{1}$$

where the linear operators $\mathcal{T}_{ij}$, $i,j=1,2$, are defined as follows.

1.

$$\mathcal{T}_{11}:AC^n\to L^n;\tag{$\mathcal{T}_{11}$}$$

$$(\mathcal{T}_{11}x)(t)=\int\limits_0^t K^1(t,s)\dot x(s)\,ds+A^1(t)x(0),\quad t\in[0,T].$$

Here the kernel $K^1(t,s)$ with its elements $k^1_{ij}(t,s)$ satisfies *the condition* $\mathcal{K}$: $k^1_{ij}(t,s)$, $i,j=1,\dots,n$, are measurable on the set $0\le s\le t\le T$ and there exists a summable nonnegative function $\kappa(\,\cdot\,)\in L^1[0,T]$ such that $|k^1_{ij}(t,s)|\le\kappa(t)$, $t\in[0,T]$, $i,j=1,\dots,n$; $(n\times n)$-matrix A^1 has elements summable on $[0,T]$.

2.

$$\mathcal{T}_{12}:FD^\nu(\mu)\to L^n;\quad (\mathcal{T}_{12}z)(t)=\sum_{\{j:\ t_j\le t\}}B^1_j(t)z(t_j),\ \ t\in[0,T], \tag{$\mathcal{T}_{12}$}$$

where elements of matrices B^1_j, $j=0,\dots,\mu$, are summable on $[0,T]$.

3.

$$\mathcal{T}_{21}:AC^n\to FD^\nu(\mu); \tag{$\mathcal{T}_{21}$}$$

$$(\mathcal{T}_{21}x)(t_i)=\int_0^{t_i}K^2_i(s)\dot{x}(s)\,ds+A^2_ix(0),\ \ i=0,1,\dots,\mu,$$

with measurable and essentially bounded on $[0,T]$ elements of matrices K^2_i and constant $(\nu\times n)$-matrices A^2_i, $i=0,1,\dots,\mu$.

4.

$$\mathcal{T}_{22}:FD^\nu(\mu)\to FD^\nu(\mu);\quad (\mathcal{T}_{22}z)(t_i)=\sum_{j=0}^{i-1}B^2_{ij}z(t_j),\ \ i=1,\dots,\mu, \tag{$\mathcal{T}_{22}$}$$

with constant $(\nu\times\nu)$-matrices B^2_{ij}.

In what follows we shall use some results from [6, 2] concerning the equation

$$\dot{x}=\mathcal{T}_{11}x+f \tag{2}$$

and the results of [1] concerning the equation

$$z=\mathcal{T}_{22}z+g. \tag{3}$$

The general solution of (2) has the form

$$x(t)=X(t)\alpha+\int_0^t C_1(t,s)f(s)\,ds,$$

with arbitrary $\alpha\in R^n$, where $X(\,\cdot\,)$ is the fundamental matrix, $C_1(\,\cdot\,,\,\cdot\,)$ is the Cauchy matrix.

As for equation (3), it has the immediate analogs of the above terms. Thus, the general solution of (3) has the representation

$$z(t_i)=Z(t_i)\beta+(C_2g)(t_i),\ \ i=1,\dots,\mu,$$

with arbitrary $\beta\in R^\nu$, where $Z(\,\cdot\,)$ is the fundamental matrix, $C_2(\,\cdot\,,\,\cdot\,)$ is the Cauchy matrix.

3 An Optimal Control Problem for a Continuous-Discrete Functional Differential System

Let us fix the initial state of the system (1):

$$x(0)=\alpha,\quad z(0)=\beta. \tag{4}$$

Next we assume that the constraints with respect to the control are formed as a system of linear inequalities:

$$Gu(t) \leqslant \gamma, \quad t \in [0,T], \tag{5}$$

where G is a given $(N \times r)$-matrix; also it is assumed that the set of all solutions to the system $Gv \leqslant \gamma$ (that is the set of admissible control values) is nonempty and bounded in R^r. Let us denote this set by $\mathcal{V}$.

As for the aim of control, it is defined with the use of a linear bounded functional $\Lambda : AC^n \times FD^\nu(\mu) \times L_2^r \to R$,

$$\Lambda(x,z,u) = l_1 x + l_2 z + \lambda u,$$

where $l_1 : AC^n \to R$, $l_2 : FD^\nu(\mu) \to R$, $\lambda : L_2^r \to R$ are linear bounded functionals.

We need to find an admissible control $u : [0,T] \to R^r$ under which the corresponding trajectory of (1) with conditions (2) brings a minimal value to the objective functional Λ. Thus we consider the optimal control problem

$$\Lambda(x,z,u) \longrightarrow \min \text{with constraints (1), (4), (5).} \tag{6}$$

Let us recall the general form of $l_1 :\ l_1 x = \psi_1 x(0) + \int\limits_0^T \varphi_1(s)\dot{x}(s)\,ds$ and $\lambda :\ \lambda u = \int\limits_0^T \lambda(s)u(s)\,ds$. Here ψ_1 is a constant $(1 \times n)$-vector, $\varphi_1(s)$ is a $(1 \times n)$-vector with elements bounded in essence, $\lambda^\perp(\,\cdot\,) \in L_2^r$. As for l_2, we put $l_2 z = \sum\limits_{i=0}^{\mu} q_i z(t_i)$ with given $(1 \times \mu)$-vectors q_i, $i = 0,\dots,\mu$.

Lemma 1. *The operator $\mathcal{T} : AC^n \to L^n$, $\mathcal{T} = \mathcal{T}_{11} + \mathcal{T}_{12}C_2\mathcal{T}_{21}$ can be represented in the form*

$$(\mathcal{T}x)(t) = \int\limits_0^t K(t,s)\dot{x}(s)\,ds + A(t)x(0), \quad t \in [0,T],$$

where the kernel $K(t,s)$ satisfies the condition $\mathcal{K}$, the columns of the matrix $A(\,\cdot\,)$ belongs to the space L^n.

Remark 1. The kernel $K(t,s)$ and the matrix A can be effectively constructed.

Lemma 2. *The functional $l : AC^n \to R$, $l = l_1 + l_2C_2\mathcal{T}_{21}$ can be represented in the form*

$$lx = \psi x(0) + \int\limits_0^T \varphi(s)\dot{x}(s)\,ds,$$

where ψ is a constant $(1 \times n)$-vector, $\varphi(s)$ is $(1 \times n)$-vector with essentially bounded elements.

Remark 2. The vectors ψ and $\varphi(s)$ can be effectively constructed.

Below we shall use the kernel $K(t,s)$ and the function $\varphi(s)$ to formulate the main result.

Now denote by $\vartheta : [0,T] \to (R^n)^*$ the solution to the integral equation

$$\vartheta(t) = \int\limits_t^T \vartheta(\tau)K(\tau,t)\,d\tau - \int\limits_t^T \varphi(\tau)K(\tau,t)\,d\tau, \quad t \in [0,T]. \tag{7}$$

The unique solvability of this equation is established in [6]. As for properties of the solution that are generated by properties of the kernel $K(t,s)$ as a function of the second argument, those are studied in [7], where in particular some conditions are formulated under which the function $\vartheta(\,\cdot\,)$ iherits the corresponding properties of $K(t,\,\cdot\,)$ (being of bounded variation, continuous, absolutely continuous). Define the functional $H : [0,T] \times (L^n)^* \times (L^n)^* \times R^r \to R$ by the equality

$$H\big(t, v(\,\cdot\,), w(\,\cdot\,), u\big) = F^*(v - w)(t) \cdot u - \lambda(t) \cdot u.$$

Here the symbol $*$ stands for adjoint spaces and operators.

Theorem. *The control $\overline{u}(t)$ solves problem* (6) *if and only if the equality*

$$H\big(t,\vartheta(\,\cdot\,),\varphi(\,\cdot\,),\overline{u}(t)\big)=\max_{u\in\mathcal{V}}H\big(t,\vartheta(\,\cdot\,),\varphi(\,\cdot\,),u\big)$$

holds almost everywhere on $[0,T]$.

Remark 3. In the case where the matrix $C(t,s)$ of the system $\dot{x}=\mathcal{T}x+f$ is known the function $\vartheta(t)$ can be written in the following explicit form:

$$\vartheta(t)=\int\limits_t^T\varphi(\tau)C'_\tau(\tau,t)\,d\tau.$$

Let us give three explicit forms of the functional H, which correspond to the following cases of F.

Case 1. $(Fu)(t)=F(t)u(t)$. For such a case we have

$$H\big(t,v(\,\cdot\,),w(\,\cdot\,),u\big)=(v(t)-w(t))\cdot F(t)\cdot u-\lambda(t)\cdot u.$$

Here the columns of $(n\times r)$-matrix $F(\,\cdot\,)$ are from L_2^n.

Case 2. $(Fu)(t)=\int\limits_0^t F(t,\tau)u(\tau)\,d\tau$. For this case, H has the representation

$$H\big(t,v(\,\cdot\,),w(\,\cdot\,),u\big)=\int\limits_t^T(v(s)-w(s))\cdot F(s,t)\,ds\cdot u-\lambda(t)\cdot u.$$

Here the kernel $F(t,\tau)$ provides the continuous action of the integral operator F from L_2^r into L^n.

Case 3. $(Fu)(t)=\begin{cases}F(t)u(t-\Delta) & \text{if } t\in[\Delta,T],\\ 0 & \text{otherwise,}\end{cases}$ where Δ, $0<\Delta<T$, is a constant delay. In such a case the functional H is defined by the equality

$$H\big(t,v(\,\cdot\,),w(\,\cdot\,),u\big)=\chi_{[0,T-\Delta]}(t)\big(v(t+\Delta)-w(t+\Delta)\big)\cdot F(t+\Delta)\cdot u-\lambda(t)\cdot u,$$

$\chi_{[0,T-\Delta]}(\,\cdot\,)$ is the characteristic function of the segment $[0,T-\Delta]$.

It should be noted that an approach to derivation of the maximum principle on the base of the variational derivatives, covering nonlinear systems with aftereffect, is thoroughly treated in [4]. Our approach is based on the use of the Cauchy matrix of the linear system and allows one to formulate the maximum principle in the terms of control only. In this case the role of the adjoint equation is played by equation (7) whose form is unified and common for all possible kinds of aftereffect in the frame of problem (6). The case of a functional differential system with continuous time only is considered in [8].

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