Sign-Constant Periodic Solutions to Second-Order Differential Equations with a Sub-Linear Non-Linearity

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We are interested in the question on the existence and uniqueness of a **non-trivial non-negative** (resp. **positive**) solution to the periodic boundary value problem

$$\boxed{u'' = p(t)u + (-1)^i q(t,u); \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega).} \tag{1_i}$$

Here, $p \in L([0,\omega])$, $q : [0,\omega] \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, and $i \in \{1,2\}$. Under a *solution* to problem (1), as usually, we understand a function $u : [0,\omega] \to \mathbb{R}$ which is absolutely continuous together with its first derivative, satisfies given equation almost everywhere and verifies periodic conditions. A solution u to problem (1) is referred as a *sign-constant solution* if there exists $i \in \{0,1\}$ such that $(-1)^i u(t) \ge 0$ for $t \in [0,\omega]$, and a *sign-changing solution* otherwise.

Definition 1. We say that the function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}^+(\omega)$ (resp. $\mathcal{V}^-(\omega)$) if for any function $u \in AC^1([0,\omega])$ satisfying

$$u''(t) \ge p(t)u(t) \;\text{ for a.e. }\; t \in [0,\omega], \quad u(0) = u(\omega), \quad u'(0) = u'(\omega),$$

the inequality

$$u(t) \ge 0 \;\text{ for }\; t \in [0,\omega] \quad \big(\text{resp. }\; u(t) \le 0 \;\text{ for }\; t \in [0,\omega]\,\big)$$

is fulfilled.

Definition 2. We say that the function $p \in L([0,\omega])$ belongs to the set $\mathcal{V}_0(\omega)$ if the problem

$$u'' = p(t)u; \quad u(0) = u(\omega), \;\; u'(0) = u'(\omega)$$

has a nontrivial sign-constant solution.

Theorem 1_1. *Let $p \in \mathcal{V}^-(\omega)$,*

$$\left.\begin{aligned}
&q(t,x) \le q_0(t,x) \;\textit{ for a.e. }\; t \in [0,\omega] \;\textit{ and all }\; x \ge x_0,\\
&x_0 \ge 0, \quad q_0 : [0,\omega] \times [x_0,+\infty[\, \to \mathbb{R} \;\textit{ is a Carathéodory function},\\
&q_0(t,\cdot) : [x_0,+\infty[\, \to \mathbb{R} \;\textit{ is non-decreasing for a.e. }\; t \in [0,\omega],\\
&\lim_{x\to+\infty} \frac{1}{x} \int_0^\omega |q_0(s,x)|\,\mathrm{d}s = 0,
\end{aligned}\right\} \tag{H_1}$$

and there exist a function $\alpha \in AC^1([0,\omega])$ *satisfying*

$$\alpha''(t) \geq p(t)\alpha(t) - q(t,\alpha(t)) \ \text{ for a.e. } t \in [0,\omega], \quad \alpha(0) = \alpha(\omega), \quad \alpha'(0) \geq \alpha'(\omega).$$

Then problem (1_1) *has at least one solution* u *such that*

$$u(t) \geq \alpha(t) \ \text{ for } t \in [0,\omega].$$

Theorem 1_2. *Let* $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$ *and*

$$\left.\begin{aligned} & |q(t,x)| \leq q_0(t,x) \ \text{ for a.e. } t \in [0,\omega] \ \text{ and all } x \geq x_0, \\ & x_0 > 0, \quad q_0 : [0,\omega] \times [x_0,+\infty[\to [0,+\infty[\ \text{ is a Carathéodory function}, \\ & q_0(t,\cdot) : [x_0,+\infty[\to [0,+\infty[\ \text{ is non-decreasing for a.e. } t \in [0,\omega], \\ & \lim_{x\to+\infty} \frac{1}{x} \int_0^\omega q_0(s,x)\,\mathrm{d}s = 0. \end{aligned}\right\} \qquad (H_2)$$

Let, moreover,

$$q(t,0) \leq 0 \ \text{ for a.e. } t \in [0,\omega] \tag{2}$$

and there exist a function $\beta \in AC^1([0,\omega])$ *satisfying*

$$\begin{gathered} \beta(t) > 0 \ \text{ for } t \in [0,\omega], \\ \beta''(t) \leq p(t)\beta(t) + q(t,\beta(t)) \ \text{ for a.e. } t \in [0,\omega], \quad \beta(0) = \beta(\omega), \quad \beta'(0) \leq \beta'(\omega). \end{gathered} \tag{3}$$

Then problem (1_2) *has at least one solution* u *such that*

$$u(t) \geq 0 \ \text{ for } t \in [0,\omega], \ \ u \not\equiv 0, \tag{4}$$

and, moreover,

$$u(t_u) \geq \beta(t_u) \ \text{ for some } t_u \in [0,\omega]. \tag{5}$$

The following example shows that, under the assumptions of Theorem 1_2, problem (1_2) may have a solution u satisfying (4) and (5), which is not positive.

Example 1. Consider the problem

$$u'' = -u + 3(1-\sin t)\sqrt{|u|}\,\operatorname{sgn} u; \quad u(0) = u(2\pi), \ \ u'(0) = u'(2\pi). \tag{6}$$

Clearly, problem (6) is a particular case of (1_2), where $\omega := 2\pi$, $p(t) := -1$ for $t \in [0,2\pi]$, and

$$q(t,x) := 3(1-\sin t)\sqrt{|x|}\,\operatorname{sgn} x \ \text{ for } t \in [0,2\pi], \ x \in \mathbb{R}.$$

It is not difficult to verify that $p \notin \mathcal{V}^-(2\pi) \cup \mathcal{V}_0(2\pi) \cup \mathcal{V}^+(2\pi)$, hypothesis (H_2) holds with $q_0(t,x) := 3(1-\sin t)\sqrt{x}$, and condition (2) is fulfilled. Moreover, one can show that there exists a function $\beta \in AC^1([0,2\pi])$ satisfying condition (3) and

$$0 < \beta(t) \leq 1 \ \text{ for } t \in [0,2\pi].$$

On the other hand, the function

$$u(t) := (1+\sin t)^2 \ \text{ for } t \in [0,2\pi]$$

is a solution to problem (6), which satisfies conditions (4) and (5), however, it is not positive.

Now we present efficient conditions guaranteeing the existence of a non-trivial sign-constant (resp. positive) solution to problem (1_i). Introduce the assumption:

$$\left.\begin{array}{l} q(t,x)\geq xg(t,x) \ \text{ for a.e. } \ t\in[0,\omega] \ \text{ and all } \ x\in\,]0,\delta[\,, \\ 0<\delta\leq+\infty, \quad g:[0,\omega]\times]0,\delta[\,\to\mathbb{R} \ \text{ is a locally Carathéodory function,} \\ g(t,\cdot):]0,\delta[\,\to\mathbb{R} \ \text{ is non-increasing for a.e. } \ t\in[0,\omega]. \end{array}\right\} \tag{G}$$

Corollary 1_1. *Let $p\in\mathcal{V}^-(\omega)$, hypotheses (H_1) and (G) be satisfied, and*

$$\lim_{x\to\delta-} g(t,x)\leq 0 \ \text{ for a.e. } \ t\in[0,\omega], \quad \lim_{x\to0+}\int\limits_0^\omega g(s,x)\,\mathrm{d}s=+\infty. \tag{7}$$

*Then problem (1_1) has at least one **positive** solution.*

Corollary 1_2. *Let $p\notin\mathcal{V}^-(\omega)\cup\mathcal{V}_0(\omega)$, $q(\cdot,0)\equiv0$, hypotheses (H_2) and (G) be satisfied, and*

$$\lim_{x\to0+}\int\limits_E g(s,x)\,\mathrm{d}s=+\infty \ \text{ for every } \ E\subseteq[0,\omega], \ \ \operatorname{meas}E>0. \tag{8}$$

*Then problem (1_2) has at least one **non-trivial non-negative** solution.*

If, in addition, $p\in\mathcal{V}^+(\omega)$ and

$$q(t,x)\geq0 \ \text{ for a.e. } \ t\in[0,\omega] \ \text{ and all } \ x\geq0, \tag{9}$$

*then problem (1_2) has at least one **positive** solution and, moreover, any solution to this problem is either positive or non-positive.*

If $p\in\mathcal{V}^+(\omega)$ in Corollary 1_2, then assumption (8) can be relaxed to

$$\lim_{x\to0+}\int\limits_0^\omega g(s,x)\,\mathrm{d}s=+\infty. \tag{10}$$

Corollary 2_2. *Let $p\in\mathcal{V}^+(\omega)$, $q(\,\cdot\,,0)\equiv0$, and hypotheses (H_2) and (G) be satisfied. Let, moreover, condition (10) hold and*

$$\lim_{x\to\delta-} g(t,x)\geq0 \ \text{ for a.e. } \ t\in[0,\omega].$$

*Then problem (1_2) has at least one **non-trivial non-negative** solution.*

*If, in addition, (9) holds, then problem (1_2) has at least one **positive** solution and, moreover, any solution to this problem is either positive or non-positive.*

The next statements show that, under the hypothesis

$$\left.\begin{array}{l} \text{for every } \ b>a>0 \ \text{ there exists } \ h_{ab}\in L([0,\omega]) \ \text{ such that} \\ h_{ab}(t)\geq0 \ \text{ for a.e. } \ t\in[0,\omega], \ \ h_{ab}\not\equiv0, \\ q(t,x)\geq h_{ab}(t) \ \text{ for a.e. } \ t\in[0,\omega] \ \text{ and all } \ x\in[a,b], \end{array}\right\} \tag{N}$$

the assumptions $p\in\mathcal{V}^-(\omega)$ and $p\notin\mathcal{V}^-(\omega)\cup\mathcal{V}_0(\omega)$ in the above-stated results are necessary for the existence of a positive solution to problem (1_1) and a non-trivial non-negative solution to problem (1_2), respectively.

Proposition 1_1. *Let hypothesis (N) hold and problem (1_1) possess a positive solution. Then $p\in\mathcal{V}^-(\omega)$.*

Proposition 1_2. *Let hypothesis (N) hold and problem (1_2) possess a non-trivial non-negative solution. Then $p \notin \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$.*

It worth mentioning that some uniqueness type results for problem (1_i) can be also proved. However, we omit here their formulation instead of which we present consequences of the general results for the following particular case of (1_i):

$$\boxed{u'' = p(t)u + (-1)^i h(t)|u|^\lambda \operatorname{sgn} u; \quad u(0) = u(\omega), \ \ u'(0) = u'(\omega),} \tag{11_i}$$

where $p, h \in L([0,\omega])$ and $\lambda \in]0,1[$. Observe that if u is a solution to problem (11_i), then the function $-u$ is its solution, as well.

Definition 3. We say that the function $p \in L([0,\omega])$ belongs to the set $\mathcal{D}_1(\omega)$ if for any $a \in [0,\omega[$, the solution u to the initial value problem

$$u'' = \widetilde{p}(t)u; \quad u(a) = 0, \ \ u'(a) = 1$$

has at most one zero in the interval $]a, a+\omega[$, where $\widetilde{p}$ is the ω-periodic extension of the function p to the whole real axis.

Remark 1. One can show that $\mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \cup \mathcal{V}^+(\omega) \subset \mathcal{D}_1(\omega)$.

Corollary 3_1. *Let $\lambda \in]0,1[$ and*

$$h(t) \geq 0 \ \text{ for a.e. } t \in [0,\omega], \ \ h \not\equiv 0. \tag{12}$$

Then the following assertions hold:

(i) *Problem (11_1) has a positive (resp. negative) solution if and only if $p \in \mathcal{V}^-(\omega)$.*

(ii) *If $p \in \mathcal{V}^-(\omega)$, then problem (11_1) has exactly three sign-constant solutions (positive, negative, and trivial).*

Corollary 3_2. *Let $\lambda \in]0,1[$ and*

$$h(t) > 0 \ \text{ for a.e. } t \in [0,\omega]. \tag{13}$$

Then the following assertions hold:

(i) *If $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, then problem (11_2) possesses only the trivial solution.*

(ii) *If $p \in \mathcal{D}_1(\omega) \setminus [\mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)]$, then problem (11_2) possesses at least three sign-constant solutions (non-trivial non-negative, non-trivial non-positive, and trivial) and no sign-changing solutions.*

(iii) *If $p \notin \mathcal{D}_1(\omega)$, then problem (11_2) has at least three sign-constant solutions (non-trivial non-negative, non-trivial non-positive, and trivial).*

In the next statement, assumption (13) appearing in Corollary 3_2 is relaxed to (12).

Corollary 4_2. *Let $\lambda \in]0,1[$ and condition (12) be fulfilled. Then the following assertions hold:*

(i) *If $p \in \mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega)$, then problem (11_2) possesses only the trivial solution.*

(ii) *If $p \in \mathcal{V}^+(\omega)$, then problem (11_2) has exactly three solutions (positive, negative, and trivial).*

(iii) *If $p \in \mathcal{D}_1(\omega) \setminus [\mathcal{V}^-(\omega) \cup \mathcal{V}_0(\omega) \cup \mathcal{V}^+(\omega)]$, then problem (11_2) has no sign-changing solutions.*

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