Precise Asymptotic Behavior of Regularly Varying Solutions of Second Order Half-Linear Differential Equations

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We consider the second order half-linear differential equation

$$(|x'|^\alpha \operatorname{sgn} x')' + q(t)|x|^\alpha \operatorname{sgn} x = 0, \tag{A}$$

under the assumption that:

(a) α is a positive constant;

(b) $q(t)$ is a continuous and integrable function on $[a,\infty)$, $a > 0$.

Let c be a constant such that

$$c \in (-\infty, E(\alpha)), \ \text{ where } \ E(\alpha) = \frac{\alpha^\alpha}{(\alpha+1)^{\alpha+1}},$$

and let λ_1, λ_2 ($\lambda_1 < \lambda_2$) denote the real roots of the equation

$$|\lambda|^{1+\frac{1}{\alpha}} - \lambda + c = 0. \tag{1}$$

It is known [2] that equation (A) possesses regularly varying solutions $x_i(t)$ such that

$$x_i \in \mathrm{RV}\,(\lambda_i^{\frac{1}{\alpha}*}), \ \ i = 1, 2,$$

if and only if

$$\lim_{t\to\infty} t^\alpha \int_t^\infty q(s)ds = c,$$

where use is made of the asterisk notation

$$u^{\gamma*} = |u|^\gamma \operatorname{sgn} u, \ \ \gamma > 0, \quad u \in \mathbf{R}.$$

A question arises: Is it possible to determine precisely the asymptotic behavior at infinity of the solutions of (A) mentioned above? It is natural to expect that the behavior of solutions would depend heavily on the rate of decay of the function

$$Q_c(t) = t^\alpha \int_t^\infty q(s)\,ds - c$$

as $t \to \infty$. The purpose of this report is to confirm the truth of this expectation by presenting some of the results, obtained in our recent paper [3], which provide explicit asymptotic formulas for regularly varying solutions of (A).

For conciseness of presentation we assume throughout that c is a nonzero constant in $(-\infty, E(\alpha))$, in which case the real roots λ_i, $i = 1, 2$, of (1) satisfy

$$0 < \lambda_1 < \lambda_2 \ \text{ if } \ c > 0, \quad \lambda_1 < 0 < \lambda_2 \ \text{ if } \ c < 0.$$

and

$$\lambda_1 < \Big(\frac{\alpha}{\alpha+1}\Big)^{\alpha} < \lambda_2 \ \text{ regardless of the sign of } \ c.$$

First we prove the following theorems which describe how the asymptotic behavior of the regularly varying solutions $x_i(t)$, $i = 1, 2$, of (A) is affected by the function $Q_c(t)$ decaying to zero as $t \to \infty$.

Theorem 1. *Suppose that there exists a positive continuous function $\phi(t)$ on $[0,\infty)$ which decreases to 0 as $t \to \infty$ and satisfies*

$$|Q_c(t)| \le \phi(t) \ \textit{ for all large } t.$$

Then, equation (A) *possesses a regularly varying solution $x_1 \in \mathrm{RV}\,(\lambda_1^{\frac{1}{\alpha}*})$ which is expressed in the form*

$$x_1(t) = \exp\bigg\{\int\limits_T^t \Big(\frac{\lambda_1 + v_1(s) + Q_c(s)}{s^\alpha}\Big)^{\frac{1}{\alpha}*}\,ds\bigg\}, \quad t \ge T, \tag{2}$$

for some $T > a$, where $v_1(t)$ satisfies

$$v_1(t) = O(\phi(t)) \ \textit{ as } \ t \to \infty. \tag{3}$$

Theorem 2. *Suppose that there exists a continuous slowly varying function $\psi(t)$ on $[0,\infty)$ which tends to 0 as $t \to \infty$ and satisfies*

$$|Q_c(t)| \le \psi(t) \ \textit{ for all large } t.$$

Then, equation (A) *possesses a regularly varying solution $x_2 \in \mathrm{RV}\,(\lambda_2^{\frac{1}{\alpha}*})$ which is expressed in the form*

$$x_2(t) = \exp\bigg\{\int\limits_T^t \Big(\frac{\lambda_2 + v_2(s) + Q_c(s)}{s^\alpha}\Big)^{\frac{1}{\alpha}*}\,ds\bigg\}, \quad t \ge T, \tag{4}$$

for some $T > a$, where $v_2(t)$ satisfies

$$v_2(t) = O(\psi(t)) \ \textit{ as } \ t \to \infty. \tag{5}$$

In the proofs of these theorems it is crucial to determine the functions $v_i(t)$ in (2) and (4) so as to satisfy (3) and (5), respectively. This can be done by deriving the integral equations for $v_i(t)$, $i = 1, 2$, via the generalized Riccati equation associated with (A) and solving them by means of the contraction mapping principle.

It is expected that the accurate asymptotic formulas for solutions $x_i(t)$, $i = 1, 2$, could be obtained from their representations (2) and (4) provided some stronger decay conditions are imposed on $Q_c(t)$. That this is indeed the case is illustrated by the following theorems.

Theorem 3. *Let $\phi(t)$ be a positive continuously differentiable function on $[0,\infty)$ which decreases to 0 as $t \to \infty$, has the property that $t|\phi'(t)|$ is decreasing and satisfies*

$$\int\limits_a^\infty \frac{\phi(t)}{t}\,dt = \infty, \quad \int\limits_a^\infty \frac{\phi(t)^2}{t}\,dt < \infty.$$

Suppose that $Q_c(t)$ is one-signed and satisfies

$$|Q_c(t)| = \phi(t) + O(\phi(t)^2), \;\; t \to \infty.$$

Then, equation (A) *possesses a regularly varying solution $x(t)$ of index $\lambda_1^{\frac{1}{\alpha}*}$ such that*

$$x(t) \sim k_1 t^{\lambda_1^{\frac{1}{\alpha}*}} \exp\bigg\{ \frac{\lambda_1^{\frac{1}{\alpha}*}}{\lambda_1(\alpha - \mu_1)} \operatorname{sgn} Q_c \int_a^t \frac{\phi(s)}{s}\, ds \bigg\}, \;\; t \to \infty,$$

for some constant $k_1 > 0$, where $\mu_1 = (\alpha + 1)\lambda_1^{\frac{1}{\alpha}*}$.

Theorem 4. *Let $\psi(t)$ be a positive continuously differentiable slowly varying function on $[0, \infty)$ which decreases to 0 as $t \to \infty$, has the property that $t|\psi'(t)|$ is slowly varying and satisfies*

$$\int_a^\infty \frac{\psi(t)}{t}\, dt = \infty, \quad \int_a^\infty \frac{\psi(t)^2}{t}\, dt < \infty.$$

Suppose that $Q_c(t)$ is one-signed and satisfies

$$|Q_c(t)| = \psi(t) + O(\psi(t)^2), \quad t \to \infty.$$

Then, equation (A) *possesses a regularly varying solution $x(t)$ of index $\lambda_2^{\frac{1}{\alpha}}$ such that*

$$x(t) \sim k_2 t^{\lambda_2^{\frac{1}{\alpha}}} \exp\bigg\{ \frac{\lambda_1^{\frac{1}{\alpha}-1}}{\alpha - \mu_2} \operatorname{sgn} Q_c \int_a^t \frac{\psi(s)}{s}\, ds \bigg\}, \;\; t \to \infty,$$

for some constant $k_2 > 0$, where $\mu_2 = (\alpha + 1)\lambda_2^{\frac{1}{\alpha}}$.

(NB) For the almost complete exposition of theory of regular variation and its applications we refer to the treatise of Bingham et al. [1]. A comprehensive survey of results up to the year 2000 on the asymptotic analysis of second order ordinary differential equations by means of regular variation can be found in the monograph of Marić [4].

References

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