Asymptotic Representations of Solutions of Differential Equations with Regularly Varying Nonlinearities

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We consider the differential equation

$$y^{(n)} = \alpha p(t) \prod_{j=0}^{n-1} \varphi_j(y^{(j)}), \tag{1}$$

where $n \geq 2$, $\alpha \in \{-1, 1\}$, $p : [a, +\infty[\to]0, +\infty[$ is a continuous function, $a \in \mathbb{R}$, $\varphi_j : \Delta Y_j \to]0; +\infty[$ is a continuous and regularly varying as $y^{(j)} \to Y_j$ function of order σ_j, $j = \overline{0, n-1}$, where ΔY_j is some one-sided neighborhood of the point Y_0, Y_0 is equal to either 0 or $\pm\infty$[1].

The set of solutions of equation (1), that is defined in some neighborhood of $+\infty$, consists of monotonous functions and their derivatives of orders till $n-1$ and falls into two classes:

1) solutions, for each of them

$$\lim_{t\to+\infty} y^{(k-1)}(t) = \begin{cases} \text{or} \ \ \pm\infty, \\ \text{or} \ \ 0 \end{cases} \qquad (k = \overline{1, n});$$

2) solutions, for each of them there exists $k \in \{1, \dots, n\}$ such that

$$y(t) = t^{k-1} \, [c + o(1)] \ \ (c \neq 0) \ \text{ as } \ t \to +\infty.$$

From the first class of solutions a sufficiently wide subclass of solutions of the equation (1) was picked out in the works of Evtukhov V. M. and Samoĭlenko A. M. [1], Klopot A. M. [2]. Asymptotic representations for this class of solutions as $t \to +\infty$ were established and necessary and sufficient conditions for the existence of these solutions were derived there.

The aim of the paper is to derive necessary and sufficient conditions for the existence of solutions of the equation (1) and more particular case, each of that for some $k \in \{1, \dots, n\}$ admits representations

$$y(t) = t^{k-1} \, [c_0 + o(1)], \quad y^{(k-1)} = c_0 + o(1) \ \ (c_0 \neq 0) \ \text{ as } \ t \to +\infty.$$

Moreover, we establish asymptotic formulas as $t \to +\infty$ for their derivatives of orders till $n-1$ and solve a question of quantity of these solutions.

Let us introduce notation for signs of numbers from neighborhoods of ΔY_j $(j = \overline{0, n-1})$.

$$\mu_j = \begin{cases} 1, & \text{if } \ Y_j = +\infty, ; \ \text{or} \ \ Y_j = 0 \ \text{ and } \ \Delta(Y_j) \ \text{ is a right neighborhood of the point } 0, \\ -1, & \text{if } \ Y_j = -\infty, \ \text{ or } \ Y_j = 0 \ \text{ and } \ \Delta(Y_j^0) \ \text{ is a left neighborhood of the point } 0. \end{cases}$$

[1]When $Y_j = \pm\infty$, here and in the sequel all signs in the neighborhood of the point ΔY_j are assumed to have the uniform sign.

Theorem 1. *For the existence of solutions of the equation* (1)*, that admit the representation*

$$y^{(n-1)} = c + o(1) \ \ (c \neq 0) \ \ as \ \ t \to +\infty,$$

it is necessary and sufficient that $c \in \Delta Y_{n-1}$ and conditions be satisfied

$$Y_{j-1} = \begin{cases} +\infty, & if \ \mu_{n-1} > 0, \\ -\infty, & if \ \mu_{n-1} < 0, \end{cases} \quad when \ \ j = \overline{1, n-1},$$

$$\int\limits_{t_0}^{+\infty} p(\tau)\varphi_0(\mu_0\tau^{n-1})\varphi_1(\mu_1\tau^{n-2})\cdots\varphi_{n-2}(\mu_{n-2}\tau)\,d\tau < +\infty,$$

where $t_0 \geq a$ is chosen so that $\frac{ct^{n-k}}{(n-k)!} \in \Delta Y_{k-1}$ $(k = \overline{1, n-1})$ for $t \geq t_0$.

Moreover, when these conditions are implemented, there exists an n-parameter family of such solutions and each of them admits the following asymptotic representations as $t \to +\infty$:

$$y^{(j-1)}(t) = \frac{ct^{n-j}}{(n-j)!}\,[1 + o(1)] \ \ (j = \overline{1, n-1}),$$

$$y^{(n-1)}(t) = c + \alpha M(c)\varphi_{n-1}(c) \int\limits_{+\infty}^{t} p(\tau)\varphi_0(\mu_0\tau^{n-1})\varphi_1(\mu_1\tau^{n-2})\cdots\varphi_{n-2}(\mu_{n-2}\tau)\,d\tau \cdot [1 + o(1)],$$

where

$$M(c) = \prod_{k=1}^{n-1} \Big|\frac{c}{(n-k)!}\Big|^{\sigma_{k-1}}.$$

Let us introduce the notation needed in the forthcoming theorem.

$$I(t) = \alpha\varphi_{n-2}(c)M(c) \int\limits_{B}^{t} p(\tau)\varphi_0(\mu_{n-2}\tau^{n-2})\varphi_1(\mu_{n-2}\tau^{n-3})\cdots\varphi_{n-3}(\mu_{n-3}\tau)\,d\tau,$$

$$M(c) = \prod_{k=1}^{n-2} \Big|\frac{c}{(n-k-1)!}\Big|^{\sigma_{k-1}}, \quad c \in \Delta Y_{n-2}, \quad \Phi(z) = \int\limits_{A}^{z} \frac{ds}{\varphi_{n-1}(s)}\,,$$

$$A = 0 \ \text{ and } \ B = +\infty, \ \text{ if } \int\limits_{y_{n-1}}^{Y_{n-1}} \frac{ds}{\varphi_{n-1}(s)} < +\infty \ \ (\sigma_{n-1} < 1, \ \ y_{n-1} \in \Delta Y_{n-1}),$$

$$A = y_{n-1} \ \text{ and } \ B = t_0, \ \text{ if } \int\limits_{y_{n-1}}^{Y_{n-1}} \frac{ds}{\varphi_{n-1}(s)} = \pm\infty \ \ (\sigma_{n-1} > 1, \ \ y_{n-1} \in \Delta Y_{n-1}).$$

Theorem 2. *Let $\sigma_{n-1} \neq 1$. For the existence of solutions of the equation* (1)*, that admit the representation*

$$y^{(n-2)}(t) = c + o(1) \ \ (c \neq 0) \ \ when \ \ t \to +\infty,$$

it is necessary and sufficient that $c \in \Delta Y_{n-2}$ and conditions be satisfied

$$Y_{n-1} = 0, \quad Y_{j-1} = \begin{cases} +\infty, & if \ \mu_{n-2} > 0, \\ -\infty, & if \ \mu_{n-2} < 0, \end{cases} \quad when \ \ j = \overline{1, n-2},$$

$$\int\limits_{t_0}^{+\infty} \Phi^{-1}(I(\tau))\,d\tau < +\infty,$$

where $t_0 \ge a$ is chosen so that

$$\frac{ct^{n-k-1}}{(n-k-1)!} \in \Delta Y_{k-1} \ \ (k = \overline{1, n-2}) \ \ \textit{for} \ \ t \ge t_0,$$

Φ^{-1} is an inverse function for Φ.

Moreover, when these conditions are implemented, there exists an n-parameter family of such solutions, if $\sigma_{n-1} < 1$, and $(n-1)$-parameter family of such solutions, if $\sigma_{n-1} < 1$, and each of them admits the following asymptotic representations as $t \to +\infty$:

$$\begin{aligned} y^{(j-1)}(t) &= \frac{ct^{n-j-1}}{(n-j-1)!}\,[1+o(1)] \ \ (j = \overline{1, n-2}), \\ y^{(n-2)}(t) &= c + \int\limits_{+\infty}^{t} \Phi^{-1}(I(s))ds\,[1+o(1)], \\ y^{(n-1)}(t) &= \Phi^{-1}(I(t))\,[1+o(1)]. \end{aligned}$$

Let us consider a particular type of equation (2)

$$y^{(n)} = \alpha p(t)\varphi_0(y)\varphi_1(y') \cdots \varphi_{n-k}(y^{(n-k)}), \tag{2}$$

where $n \ge 2$, $\alpha \in \{-1, 1\}$, $k \in \{1, \dots, n\}$, $p : [a, +\infty[\to]0, +\infty[$ is a continuous function, $a \in \mathbb{R}$, $\varphi_j : \Delta Y_j \to]0; +\infty[$ is a continuous and regularly varying as $y^{(j)} \to Y_j$ function of order σ_j, $j = \overline{0, n-k}$, where ΔY_j is some one-sided neighborhood of the point Y_j, Y_j is equal to either 0 or $\pm\infty$.

Theorem 3. *For the existence of solutions of the equation (2), that admit the representation as $i \in \{1, \dots, k\}$:*

$$y^{(n-k)}(t) = \frac{ct^{i-1}}{(i-1)!}\,[1+o(1)] \ ; \ \ (c \ne 0) \ \ \textit{as} \ \ t \to +\infty,$$

it is necessary and sufficient that $c \in \Delta Y_{n-k}$ and conditions be satisfied

$$Y_{j-1} = \begin{cases} +\infty, & \textit{if} \ \ \mu_{n-k} > 0, \\ -\infty, & \textit{if} \ \ \mu_{n-k} < 0, \end{cases} \quad \textit{when} \ \ j = \overline{1, n-k}, \ \ \textit{if} \ \ i = 1;$$

$$Y_{j-1} = \begin{cases} +\infty, & \textit{if} \ \ \mu_{n-k} > 0, \\ -\infty, & \textit{if} \ \ \mu_{n-k} < 0, \end{cases} \quad \textit{when} \ \ j = \overline{1, n-k+1}, \ \ \textit{if} \ \ i > 1;$$

$$\int\limits_{t_0}^{+\infty} \int\limits_{t_{k-i}}^{+\infty} \cdots \int\limits_{t_1}^{+\infty} p(\tau)\varphi_0(\mu_0 \tau^{n-k+i-1}) \cdots \varphi_{n-k}(\mu_{n-k}\tau^{i-1})\, dt_1 \cdots dt_{k-i}\, d\tau < +\infty,$$

where $t_0 \ge a$ is chosen so that

$$\frac{ct^{n-k+i-j}}{(n-k+i-j)!} \in \Delta Y_{j-1} \ \ (j = \overline{1, n-k-1}) \ \ \textit{for} \ \ t \ge t_0.$$

Moreover, when these conditions are implemented, there exists an n-parameter family of such solutions and each of them admits the following asymptotic representations as $t \to +\infty$:

$$\begin{aligned} y^{(j-1)}(t) &= \frac{ct^{n-k+i-j}}{(n-k+i-j)!}\,[1+o(1)] \ \ (j = \overline{1, n-k+i-1}), \\ y^{(n-k+i-1)}(t) &= c + \alpha M(c) W_{k-i+1}(t)[1+o(1)], \\ y^{(j)}(t) &= \alpha M(c) W_{n-j}(t)[1+o(1)] \ \ (j = \overline{n-k+i, n-1}), \end{aligned}$$

where

$$M(c)=\prod_{j=1}^{n-k+1}\Big|\frac{c}{(n-k+i-j)!}\Big|^{\sigma_{k-1}},\ \ c\in\Delta Y_{n-k},\quad W_j(t)=\int\limits_{+\infty}^{t}W_{j-1}(s)\,ds\ \ (j=\overline{1,k-i+1}),$$

$$W_0(t)=p(t)\varphi_0(\mu_0t^{n-k+i-1})\varphi_1(\mu_1t^{n-k+i-2})\cdots\varphi_{n-k}(\mu_{n-k}t^{i-1}).$$

References

[1] V. M. Evtukhov and A. M. Samoĭlenko, Asymptotic representations of solutions of nonautonomous ordinary differential equations with regularly varying nonlinearities. (Russian) *Differ. Uravn.* **47** (2011), No. 5, 628–650; translation in *Differ. Equ.* **47** (2011), No. 5, 627–649.

[2] A. M. Klopot, Asymptotic representations of solutions of n-order differential equations with regularly varying nonlinearitites. *Dissert. Kand. Phis.-Mat. Nauk*, 2015, 148 pp.