On One Two-Dimensional Nonlinear Integro-Differential Equation

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As it is known the magnetic field diffusion process in the medium can be modeled by Maxwell's system of partial differential equations [1]. Assume that coefficients of thermal heat capacity and electroconductivity of the substance depend on temperature. In this case, as it is shown in [2], the system of Maxwell's equation can be reduced to the following integro-differential form

$$\frac{\partial H}{\partial t} = -\operatorname{rot}\bigg[a\bigg(\int_0^t |\operatorname{rot} H|^2\,d\tau\bigg)\operatorname{rot} H\bigg], \tag{1}$$

where $H = (H_1, H_2, H_3)$ is a vector of the magnetic field and function $a = a(S)$ is defined for $S \in [0, \infty)$.

In the work [3] some generalization of equations of type (1) is proposed. In particular, if the temperature is kept constant throughout the material, the same process of penetration of a magnetic field into a substance can be rewritten in the following integro-differential form [3]:

$$\frac{\partial H}{\partial t} = a\bigg(\int_0^t \int_\Omega |\operatorname{rot} H|^2\,dx\,d\tau\bigg)\Delta H, \tag{2}$$

where $x \in \Omega \subset R^3$.

Note that integro-differential parabolic models of (1) and (2) type are complex and still yield to the investigation only for special cases (see, for example, [2], [4]–[17] and references therein). Investigations mainly are done for one-dimensional case, i.e., when components of magnetic field H depend on one space variable.

The existence of a weak solution to the first boundary value problem for the one component magnetic field and one dimensional spatial version for the case $a(S) = 1 + S$ and uniqueness results for some general cases of model (1) were proved in [2]. The same questions for model (2) has been discussed in [8].

The theorems and discussions of a large time behavior to the solutions of the initial-boundary value problems for the one-dimensional analog of (2) type models for the different cases of function $a = a(S)$ are studied in [4], [8]–[14], [16]. The multidimensional case for (1) type model is considered in [6]. The questions of numerical solution of corresponding initial-boundary value problems for (2) type models are discussed in [7], [12]–[17].

Purpose of this note is to study asymptotic behavior as $t \to \infty$ of a solution of the Dirichlet problem for model (2) in one component magnetic field and two-dimensional spatial case. Assume that the magnetic field has the following form $H = (0, 0, U)$ and $U = U(x, y, t)$. Then we have

$$\operatorname{rot} H = \Big(\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial x}, 0\Big)$$

and equation (2) takes the following form

$$\frac{\partial U}{\partial t} = a(S)\Big(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2}\Big), \ \ (x, t) \in Q = \Omega \times (0, \infty), \tag{3}$$

where
$$S(t)=\int_0^t\int_\Omega\bigg[\Big(\frac{\partial U}{\partial x}\Big)^2+\Big(\frac{\partial U}{\partial y}\Big)^2\bigg]\,dx\,dy\,d\tau, \tag{4}$$
and $\Omega=(0,1)\times(0,1)$.

In the domain Q, let us consider the following initial-boundary value problem for equation (3), (4):
$$U(x,y,t)=0,\quad (x,y)\in\partial\Omega,\ \ t\geq 0, \tag{5}$$
$$U(x,y,0)=U_0(x,y),\quad (x,y)\in\overline{\Omega}, \tag{6}$$
where $U_0=U_0(x,y)$ is a given function.

Recall the L_2-inner product and norm:
$$(u,v)=\int_\Omega u(x,y)v(x,y)\,dx\,dy,\quad \|u\|=(u,u)^{1/2}.$$

The following statements take place.

Theorem 1. *If $a(S)=(1+S)^p$, $p>0$; $U_0\in H_0^1(\Omega)$, then for the solution of problem* (3)–(6) *the following estimate is true*
$$\Big\|\frac{\partial U}{\partial x}\Big\|^2+\Big\|\frac{\partial U}{\partial y}\Big\|^2\leq C\exp(-2t).$$

Here and below we use usual Sobolev spaces $H^k(\Omega)$ and $H_0^k(\Omega)$ and constant C which denotes various positive values independent of t.

Note that Theorem 1 gives exponential stabilization of the solution of problem (3)–(6) in the norm of the space $H^1(\Omega)$.

Theorem 2. *If $a(S)=(1+S)^p$, $p>0$; $U_0\in H_0^1(\Omega)\cap H^2(\Omega)$, then for the solution of problem* (3)–(6) *the following estimate is true*
$$\Big\|\frac{\partial U(x,t)}{\partial t}\Big\|\leq C\exp\Big(-\frac{t}{2}\Big).$$

The algorithm of an approximate solution is constructed by using of which numerous numerical experiments for problem (3)–(6) with different kind of initial-boundary value problems are carried out. Results of numerical experiments agree with the theoretical ones obtained in Theorems 1 and 2.

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