On Boundary Value Problems with the Condition at Infinity for Systems of Higher Order Nonlinear Differential Equations

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In the interval $\mathbb{R}_+ = [0, +\infty[$, we consider the problem on the existence of a solution of the nonlinear differential system

$$u^{(m)} = f_1(t, v, \dots, v^{(n-1)}), \quad v^{(n)} = f_2(t, u, \dots, u^{(m-1)}), \tag{1}$$

satisfying the boundary conditions

$$\begin{gathered} u^{(i-1)}(0) = \varphi_i\big(v^{(n-1)}(0)\big) \;\; (i = 1, \dots, m), \quad v^{(k-1)}(0) = \psi_k\big(v^{(n-1)}(0)\big) \;\; (k = 1, \dots, n-1), \\ \liminf_{t\to+\infty} |v^{(n-1)}(t)| = 0. \end{gathered} \tag{2}$$

Here $m \geq 1$, $n \geq 2$, and $f_1 : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}$, $f_2 : \mathbb{R}_+ \times \mathbb{R}^m \to \mathbb{R}$, $\varphi_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, \dots, m)$, $\psi_k : \mathbb{R} \to \mathbb{R}$ $(k = 1, \dots, n-1)$ are continuous functions.

Problem (1), (2) is interesting because its different particular cases arise in the oscillation theory (see, e.g., [1, 2]). Nevertheless, in the general case this problem is not studied yet. We have established sufficient conditions for the solvability and unique solvability of that problem. In particular, the following theorems are proved.

Theorem 1. *Let there exist a positive constant r and continuous functions $h_0 : \mathbb{R}_+ \to \mathbb{R}_+$, $h_k : \mathbb{R}_+ \to \mathbb{R}_+$ $(k = 1, \dots, n)$ such that*

$$\big|f_1(t, x_1, \dots, x_n)\big| \leq h_0(t)\Big(1 + \sum_{i=1}^{n} |x_i|\Big) \;\; \text{for } t \in \mathbb{R}_+, \;\; (x_1, \dots, x_n) \in \mathbb{R}^n,$$

$$f_1(t, x_1, \dots, x_n)\operatorname{sgn}(x_1) \geq \sum_{k=1}^{n} h_k(t)|x_k| \;\; \text{for } t \in \mathbb{R}_+, \;\; x_i \operatorname{sgn}(x_1) \geq r \;\; (i=1, \dots, n-1), \;\; x_n x_1 > 0,$$

$$\big|f_2(t, x_1, \dots, x_m)\big| \leq h_0(t)\Big(1 + \sum_{i=1}^{m} |x_i|\Big) \;\; \text{for } t \in \mathbb{R}_+, \;\; (x_1, \dots, x_m) \in \mathbb{R}^m,$$

$$f_2(t, x_1, \dots, x_m)\operatorname{sgn}(x_1) \geq 0 \;\; \text{for } t \in \mathbb{R}_+, \;\; x_i \operatorname{sgn}(x_1) \geq r \;\; (i = 1, \dots, m),$$

and

$$\liminf_{|x|\to+\infty} \varphi_i(x)\operatorname{sgn}(x) > r \;\; (i = 1, \dots, n), \quad \liminf_{|x|\to+\infty} \psi_k(x)\operatorname{sgn}(x) > r \;\; (k = 1, \dots, n-1).$$

If, moreover,

$$\int_0^\infty \Big(\sum_{k=1}^{m} t^{n-k} h_k(t)\Big)\, dt = +\infty \;\; (k = 1, \dots, n), \tag{3}$$

then problem (1), (2) has at least one solution.

Theorem 2. *Let the functions f_i $(i=1,2)$ have continuous partial derivatives in the phase variables,*

$$f_i(t,0,\dots,0)=0 \ \text{ for } \ t\in\mathbb{R}_+ \ \ (i=1,2),$$

and let there exist continuous functions $h_k:\mathbb{R}_+\to\mathbb{R}_+$ $(k=0,\dots,n)$ such that

$$h_k(t)\le \frac{\partial f_1(t,x_1,\dots,x_n)}{\partial x_k}\le h_0(t) \ \text{ for } \ t\in\mathbb{R}_+, \ \ (x_1,\dots,x_n)\in\mathbb{R}^n \ \ (k=1,\dots,n),$$

$$0\le \frac{\partial f_2(t,x_1,\dots,x_m)}{\partial x_k}\le h_0(t) \ \text{ for } \ t\in\mathbb{R}_+, \ \ (x_1,\dots,x_m)\in\mathbb{R}^m \ \ (k=1,\dots,m).$$

Let, moreover, h_k $(k=1,\dots,n)$ satisfy condition (3)*, while φ_i $(i=1,\dots,m)$ and ψ_k $(k=1,\dots,n-1)$ be nondecreasing functions such that*

$$\liminf_{|x|\to+\infty}\varphi_i(x)\operatorname{sgn}(x)>0 \ \ (i=1,\dots,m), \qquad \liminf_{|x|\to+\infty}\psi_k(x)\operatorname{sgn}(x)>0 \ \ (k=1,\dots,n-1).$$

Then problem (1), (2) *has one and only one solution.*

Acknowledgement

Supported by the Shota Rustaveli National Science Foundation (Project # FR/317/5-101/12).

References

[1] I. Kiguradze, Some singular boundary value problems for ordinary differential equations. (Russian) *Tbilisi University Press, Tbilisi*, 1975.

[2] I. T. Kiguradze and T. A. Chanturia, Asymptotic properties of solutions of nonautonomous ordinary differential equations. Translated from the 1985 Russian original. Mathematics and its Applications (Soviet Series), 89. *Kluwer Academic Publishers Group, Dordrecht*, 1993.