On One Boundary Value Problem with the Condition at Infinity, Arising in the Oscillation Theory

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In the infinite interval $\mathbb{R}_+ = [0, +\infty[$, we consider the $(n \geq 2)$-th order differential equation

$$u^{(n)}(t) = f\Big(t, u(t), \dots, u^{(n-1)}(t), u(\tau_1(t)), \dots, u^{(n-1)}(\tau_n(t))\Big) \tag{1}$$

with the boundary conditions

$$u^{(i-1)}(0) = \varphi_i\big(u^{(n-1)}(0)\big) \ \ (i = 1, \dots, n-1), \quad \liminf_{t \to +\infty} |u^{(n-1)}(t)| < +\infty, \tag{2}$$

where $f : \mathbb{R}_+ \times \mathbb{R}^{2n} \to \mathbb{R}$, $\varphi_i : \mathbb{R} \to \mathbb{R}$ $(i = 1, \dots, n-1)$ and $\tau_k : \mathbb{R}_+ \to \mathbb{R}_+$ $(k = 1, \dots, n)$ are continuous functions and

$$0 \leq \tau_k(t) < t \ \text{ for } \ t > 0, \quad \lim_{t \to +\infty} \tau_k(t) = +\infty \ \ (k = 1, \dots, n). \tag{3}$$

Problems of the type (1), (2) arise in the oscillation theory when studying the existence of proper oscillatory solutions of differential and functional differential equations having the property B (see, e.g., [1–3]).

We have found conditions guaranteeing, respectively, the solvability and unique solvability of problem (1), (2). In particular, the following theorems are proved.

Theorem 1. *Let there exist a continuous function $g : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ and a positive constant ρ such that*

$$\begin{aligned} &\big|f(t, x_1, \dots, x_n, y_1, \dots, y_n)\big| \leq \\ &\qquad \leq g(t, y_1, \dots, y_n)\Big(1 + \sum_{k=1}^{n} |x_k|\Big) \ \text{ for } \ t \in \mathbb{R}_+, \ \ (x_1, \dots, x_n, y_1, \dots, y_n) \in \mathbb{R}^{2n}, \end{aligned} \tag{4}$$

$$f(t, x_1, \dots, x_n, y_1, \dots, y_n)x_1 \geq 0 \ \text{ for } \ t \in \mathbb{R}_+, \ \ x_k \operatorname{sgn}(x_1) \geq \rho, \ \ y_k \operatorname{sgn}(y_1) \geq \rho \ \ (k = 1, \dots, n) \tag{5}$$

and

$$\liminf_{|x| \to +\infty} \varphi_i(x) \operatorname{sgn}(x) > \rho \ \ (i = 1, \dots, n-1). \tag{6}$$

Then problem (1), (2) *has at least one solution.*

Theorem 2. *Let the function f be nondecreasing and locally Lipschitz in the last $2n$ arguments and along with* (4), (5) *satisfy the condition*

$$\int_0^{+\infty} \big|f(t, t^{n-1}x, \dots, x, \tau^{n-1}(t)x, \dots, x)\big| \, dt = +\infty \ \text{ for } \ x \neq 0.$$

If, moreover, φ_i $(i = 1, \dots, n)$ are nondecreasing functions satisfying inequalities (6)*, then problem* (1), (2) *has one and only one solution.*

As examples, we consider the differential equations

$$u^{(n)}(t)=\sum_{k=1}^{n} p_k(t)\big|u^{(k-1)}(\tau_k(t))\big|^{\lambda_k}u^{(k-1)}(t)+q(t), \tag{7}$$

$$\begin{aligned}u^{(n)}(t)=&\sum_{k=1}^{n} p_{1k}(t)\big|u^{(k-1)}(\tau_k(t))\big|^{\lambda_{1k}}\operatorname{sgn}\big(u^{(k-1)}(\tau_k(t))\big)+\\ &+\sum_{k=1}^{n} p_{2k}(t)\big(1+|u^{(k-1)}(t)|\big)^{-\lambda_{2k}}u^{(k-1)}(t)+q(t)\end{aligned} \tag{8}$$

with the boundary conditions

$$u^{(i-1)}(0)=\alpha_i|u^{(n-1)}(0)|^{\mu_i}\operatorname{sgn}\big(u^{(n-1)}(0)\big)+\beta_i \ \ (i=1,\dots,n-1), \quad \liminf_{t\to+\infty}|u^{(n-1)}(t)|<+\infty, \tag{9}$$

where

$$\lambda_k>0,\ \ \lambda_{1k}\geq 1,\ \ 0\leq\lambda_{2k}\leq 1\ \ (k=1,\dots,n),$$
$$\alpha_i>0,\ \ \mu_i>0,\ \ \beta_i\in\mathbb{R}\ \ (i=1,\dots,n),$$

$p_k:\mathbb{R}_+\to\mathbb{R}_+$, $p_{ik}:\mathbb{R}_+\to\mathbb{R}_+$ $(i=1,2;\ k=1,\dots,n)$, $q:\mathbb{R}_+\to\mathbb{R}$ are continuous functions, while $\tau_k:\mathbb{R}_+\to\mathbb{R}_+$ $(k=1,\dots,n)$ are functions satisfying conditions (3).

Theorems 1 and 2 imply the following proposition.

Corollary 1. *If*

$$|q(t)|\leq r\sum_{k=1}^{m} p_k(t) \ \ \textit{for} \ \ t\in\mathbb{R}_+,$$

where $r=const>0$, then problem (7), (9) *has at least one solution.*

Corollary 2. *If*

$$|q(t)|\leq r\sum_{k=1}^{n}\big(p_{1k}(t)+p_{2k}(t)\big) \ \ \textit{for} \ \ t\in\mathbb{R}_+$$

and

$$\int_0^{+\infty}\sum_{k=1}^{n}\big(p_{1k}(t)\tau^{(n-k)\lambda_{1k}}(t)+p_{2k}(t)t^{(1-\lambda_{2k})(n-k)}\big)\,dt=+\infty, \qquad \int_0^{+\infty}|q(t)|\,dt<+\infty,$$

then problem (8), (9) *has one and only one solution.*

Acknowledgement

Supported by the Shota Rustaveli National Science Foundation (Project # FR/317/5-101/12).

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