A Complete Description of The Largest Lyapunov Exponent of Linear Differential Systems with Parameter-Multiplier as Function of Parameter

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Consider the n-dimensional ($n \geq 2$) linear system of differential equations

$$\mathrm{d}x/\mathrm{d}t = A(t)x, \ \ x \in \mathbb{R}^n, \ \ t \geq 0, \tag{1}$$

with piecewise continuous on the half-line $t \geq 0$ coefficient matrix $A(\,\cdot\,)\colon [0,+\infty) \to \operatorname{End}\mathbb{R}^n$. Denote the class of all such systems by $\mathcal{M}_n^*$. We identify the system (1) and it's coefficient matrix and therefore write $A \in \mathcal{M}_n^*$. Along with the system (1) we consider the one-parameter family

$$\mathrm{d}x/\mathrm{d}t = \mu A(t)x, \quad x \in \mathbb{R}^n, \ \ t \geq 0, \tag{2}$$

of linear differential systems with a parameter-multiplier $\mu \in \mathbb{R}$. Denote by $\lambda_1(\mu A) \leq \cdots \leq \lambda_n(\mu A)$ the Lyapunov exponents [1, p. 34], [2, p. 63] of the system (2).

V. I. Zubov in [3, p. 408; Problem 1] set the following problem: find out how the Lyapunov exponents of the systems (1) and (2) are related. For every $A \in \mathcal{M}_n^*$ we consider the exponent $\lambda_i(\mu A)$ as function of variable $\mu \in \mathbb{R}$ and call it the i-th Lyapunov exponent of the family (2). Emphasize that in [3] in the formulation of the problem it is not necessary that the coefficient matrix of (1) is bounded. Therefore the exponent $\lambda_i(\mu A)$, $i = \overline{1,n}$, can take improper values $-\infty$ and $+\infty$. Hence the function $\lambda_i(\mu A)$ is a mapping $\mathbb{R} \to \overline{\mathbb{R}}$ where $\overline{\mathbb{R}} = \mathbb{R} \sqcup \{-\infty, +\infty\}$.

Zubov problem is equivalent to the following: for every $i = \overline{1,n}$ give a complete description of the set of i-th Lyapunov exponents of families (2), i.e. the set $\mathcal{L}_i^n \stackrel{\text{def}}{=} \{\lambda_i(\mu A) \mid \ A \in \mathcal{M}_n^*\}$ of functions $\lambda_i(\mu A)\colon \mathbb{R} \to \overline{\mathbb{R}}$. In the present report this problem is solved for the largest Lyapunov exponent, i.e. a complete description of the set $\mathcal{L}_n^n$ is given for every integer $n \geq 2$.

Note that for parametric families of linear differential systems

$$\mathrm{d}x/\mathrm{d}t = A(t,\mu)x, \ \ x \in \mathbb{R}^n, \ \ t \geq 0, \tag{3}$$

with continuous in the variables t, μ and bounded on the half-line $t \geq 0$ for every fixed $\mu \in \mathbb{R}$ coefficient matrix $A(t,\mu)\colon [0,+\infty) \times \mathbb{R} \to \operatorname{End}\mathbb{R}^n$, a similar problem is solved in [4]. It is proved that for every $i = \overline{1,n}$ a function $\lambda(\,\cdot\,)\colon \mathbb{R} \to \mathbb{R}$ is the i-th Lyapunov exponent (considered as a function of $\mu \in \mathbb{R}$) of some family (3) if and only if $\lambda(\,\cdot\,)$ belongs to the Baire class $({}^*, G_\delta)$ and have an upper semicontinuous minorant. In the paper [4] it is proved that this result holds in a more general situation – for the Lyapunov exponents of families of morphisms of Millionshchikov bundles.

Recall that a real-valued function is referred to as a function of the class $({}^*, G_\delta)$ [5, p. 223–224] if for each $r \in \mathbb{R}$ the preimage of the interval $[r,+\infty)$ under the mapping f is a G_δ-set, i.e. can be represented as a countable intersection of open sets. Consider $\overline{\mathbb{R}}$ with a natural (order) topology, so that $\overline{\mathbb{R}}$ is homeomorphic to the interval $[-1,1]$. Choose such a homeomorphism $\ell\colon \overline{\mathbb{R}} \to [-1,1]$ in a standard way: $\ell(x) = \frac{x}{|x|+1}$ if $x \in \mathbb{R}$, and $\ell(x) = \operatorname{sgn}(x)$ if $x = \pm\infty$. Since the mapping ℓ performs an order-preserving homeomorphism between $\overline{\mathbb{R}}$ and $[-1,1]$, we say that a function $f\colon \mathbb{R} \to \overline{\mathbb{R}}$ belongs to the Baire class $\mathcal{K}$ if the composition $\ell \circ f$ belongs to the class $\mathcal{K}$. This definition is equivalent to the definition [6, p. 382, 401] of Baire classes of mappings between metric spaces.

Slightly modifying proofs of [4] one can get that similar result holds true for the generalized Lyapunov exponents which implies that for every $i = \overline{1,n}$ a function $\lambda(\,\cdot\,)\colon \mathbb{R} \to \overline{\mathbb{R}}$ is the i-th Lyapunov exponent (considered as a function of $\mu \in \mathbb{R}$) of some family (3) with not necessary bounded coefficients if and only if $\lambda(\,\cdot\,)$ belongs to the Baire class $({}^*, G_\delta)$.

Despite the fact that the dependence on the parameter in the families (2) is linear, the description of the largest Lyapunov exponents of families (2) is similar to the description of the largest Lyapunov exponents in the general case of families (3).

A partial solution to the Zubov problem was announced in report [7]. In the paper [8] it was proved that conditions 1) – 4) of the theorem below are necessary. In [8] it was also proved that conditions 1) – 3) are sufficient under the assumption that there exists such a real number b that the inequality $f(\mu) \geq b\mu$ holds for all $\mu \in \mathbb{R}$. In the general case the theorem below gives a complete description of the set $\mathcal{L}_n^n$ for an arbitrary integer $n \geq 2$.

Theorem. *A function $f\colon \mathbb{R} \to \overline{\mathbb{R}}$ belongs to the class $\mathcal{L}_n^n = \{\lambda_n(\mu A)\colon \mathbb{R} \to \overline{\mathbb{R}} \mid A \in \mathcal{M}_n^*\}$ for an arbitrary $n \geq 2$ if and only if it fits the next four conditions:*

1) *f belongs to $({}^*, G_\delta)$ Baire class;*

2) *$f(0) = 0$;*

3) *f is nonnegative on some real semiaxis;*

4) *if f is not identically equal to $+\infty$ on any semiaxis, then there exists such a real number b that the inequality $f(\mu) \geq b\mu$ holds for all $\mu \in \mathbb{R}$.*

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