Stability of Linear Stochastic Difference Equations with Delay

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Stochastic difference equations were truly introduced in [3]. Stability of these equations is an important problem which has not been comprehensively studied yet. Some results can be found in [2, 4, 9–12]. Stochastic functional difference equations were introduced in [8] and studied further in [13]. Stability of difference equations with a random delay was studied in [6].

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a stochastic basis satisfying usual assumptions (see e.g. [7]). In what follows we assume that $\mathcal{B}_i$, $i = 2, \dots, m$ are independent standard scalar Wiener processes; E is the expectation with respect to the probability measure P; $|\cdot|$ is a fixed norm in R^n; $\|\cdot\|$ is the norm of an $n\times n$-matrix, which is consistent with the chosen vector norm in R^n; N is the set of all natural numbers; $N_+ = \{0\}\cup N$; Z is the set of all integers.

For given $1 \leq p < \infty$, $h > 0$ the number c_p^h is the universal constant for which the following inequalities are satisfied

$$E\bigg|\int\limits_t^{t+h} \varphi(s)\,d\mathcal{B}(s)\bigg|^{2p} \leq c_p^h E \int\limits_t^{t+h} |\varphi(s)|^{2p}\,ds. \tag{1}$$

The inequalities should be valid for any $t \geq 0$, any $\mathcal{F}_t$-adapted stochastic process φ and a standard scalar Wiener process $\mathcal{B}$. In [7, p. 39], these constants are defined (up to a change of the notation) as $c_p^h = p^p(2p-1)^p h^{p-1}$ for $p > 1$ and $c_1 = 1$ for $p = 1$. The Burkholder–Davis–Ghandy inequalities give the estimates which are independent of h (see e.g. [7, p. 40] where p should be replaced with $2p$).

Below we consider the following stochastic difference equations:

(a) *The linear ordinary difference Itô equation*

$$x(s+1) = x(s) + A_1(s)x(s)h + \sum_{i=2}^m A_i(s)x(j)\big(\mathcal{B}_i((s+1)h) - \mathcal{B}_i(sh)\big) \ \ (s\in N_+), \tag{2}$$

where $x(s)$ is a $\mathcal{F}_s$-measurable, n-dimensional random variable for any $s\in N_+$, h is a positive number, $A_i(s)$ is an $n\times n$-matrix, whose entries are $\mathcal{F}_s$-measurable random variables for any $i = 1, \dots, m$, $s \in N_+$.

(b) *The linear difference Itô equation with delay*

$$\begin{aligned} x(s+1) = x(s) &+ \sum_{j=-\infty}^{s} A_1^2(s,j)x(j)h + \\ &+ \sum_{i=2}^m \sum_{j=-\infty}^{s} A_i^2(s,j)x(j)\big(\mathcal{B}_i((s+1)h) - \mathcal{B}_i(sh)\big) \ \ (s\in N_+), \\ x(j) = \varphi(j) &\ \ (j<0), \end{aligned} \tag{3}$$

where $x(s)$ is a $\mathcal{F}_s$-measurable, n-dimensional random variable for any $s \in N_+$, h is a positive number, $A_i^2(s,j)$ is an $n \times n$-matrix, whose entries are $\mathcal{F}_s$-measurable random variables for any $s \in N_+$, $j = -\infty, \dots, s$, $i = 1, \dots, m$, $\varphi(j)$ $(j < 0)$ is a $\mathcal{F}_0$-measurable random variable.

Note that the equation (2) is a particular case of the equation (3). Below we therefore formulate the definitions and results in terms of (3), only.

A solution of the equation (3) is a sequence of n-dimensional and $\mathcal{F}_s$-measurable random variables $x(s)$ $(s \in Z)$, which satisfies (3) P-almost everywhere. More precisely, $x(s)$ satisfies the difference equation for $s \in N_+$ and coincides with $\varphi(s)$ for $s < 0$. Thus the only degree of freedom of the solution of (3) is its initial value $x(0) = x_0$ at $s = 0$.

Note that for any $\mathcal{F}_0$-measurable initial value x_0, the solution of (3) always exists, and it is unique up to the natural P-equivalence. Moreover, this solution is a $\mathcal{F}_s$-adapted discrete stochastic process $x : Z \times \Omega \to R^n$. Restricted to the set N_+, this solution will be denoted by $x_\varphi(s, x_0)$, $s \in N_+$.

Definition 1. The trivial solution of the equation (3) is called p-stable with respect to the initial data (φ and x_0) if for any $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that $\mathbb{E}|x_0|^p + \operatorname*{vrai\,sup}_{j<0} \mathbb{E}|\varphi(j)|^p) < \eta$ implies $E|x(s,x_0)|^p \le \varepsilon$ for all $s \in N_+$.

If, in addition, $\mathbb{E}|x_\varphi(s,x_0)|^p \to 0$ as $s \to \infty$, then the trivial solution is called asymptotically p-stable.

The first result concerns asymptotic stability of the ordinary difference equation (2).

Theorem 1. *Assume that $A_i(s) = a_i$, $i = 1, \dots, m$ for $s \in N_+$.*

If now

$$-1 < a_1 h < 0, \;\; c_p^h \sum_{i=2}^{m} |a_i| < -a_1 h^{1/2},$$

then the equation (2) *is asymptotically* $2p$*-stable with respect to initial data.*

The second result applies to the vector equation (3). However, it does not guarantee asymptotic stability.

Theorem 2. *Assume that there exist positive numbers $a_i(s,j)$, $i = 1, \dots, n$, $s \in N_+$, $j = -\infty, \dots, s$ such that the coefficients in* (3) *satisfy*

$$\|A_i(s,j)\| \le a_i(s,j), \;\; i = 1, \dots, m, \;\; s \in N_+, \;\; j = -\infty, \dots, s$$

P-almost everywhere,

$$\sum_{\tau=0}^{\infty} \sum_{j=-\infty}^{-1} a_i(\tau, j) < \infty \;\; (i = 1, \dots, m)$$

and

$$\overline{c} \overset{def}{=} \sum_{\tau=0}^{\infty} \Big(\sum_{j=0}^{\tau} a_1(\tau,j) h + c_p^h \sum_{i=2}^{m} \sum_{j=0}^{\tau} a_i(\tau,j) h^{1/2} \Big) < 1.$$

Then the trivial solution of the equation (3) *is* $2p$*-stable with respect to the initial data.*

The idea of the proofs.

The proofs of the theorems are based on Azbelev's W-transform of the equations (2) and (3), respectively (see e. g. [1]). The transform is designed in a special manner with the help of the so-called "reference equation". Usually, the latter is an equation which already possesses the desired asymptotic properties, but which is simpler than the equation to be studied. The W-method works if the integral operator, which results from the substitution of the solutions of the reference equations into the given equation, is invertible.

Applying this idea, we first of all introduce two spaces of discrete stochastic processes:

1) d^n is the linear space of all possible solutions of the difference equation (3);

2) l^n is the linear space of all sequences of $m \times n$-matrices $H(s)(s \in N_+)$, with the entries being $\mathcal{F}_s$-measurable random variables.

We will also need the following operator equation constructed from the equation (3):

$$x(s+1) = x(s) + [(Vx)(s) + f(s)]Z(s) \ \ (s \in N_+), \tag{4}$$

where

$$\begin{aligned}
(Vx)(s) &= \Big(\sum_{j=0}^{s} A_1(s,j)x(j), \sum_{j=0}^{s} A_2(s,j)x(j), \dots, \sum_{j=0}^{s} A_m(s,j)x(j) \Big) \ \ (s \in N_+),\\
f(s) &= \big(f_1(s), f_2(s), \dots, f_m(s)\big) \ \ (s \in N_+),\\
Z(s) &= \Big(h, \big(\mathcal{B}_2((s+1)h) - \mathcal{B}_2(sh)\big), \dots, \big(\mathcal{B}_m((s+1)h) - \mathcal{B}_m(sh)\big)\Big) \ \ (s \in N_+).
\end{aligned}$$

Here $f \in l^n$. Let us note that the initial function $\varphi(s)$ from (3) is in this representation included in the equation (4) as a special case of f, see the formula (8) below and [1,5] for further details. This trick gives us opportunity to study stability with respect to φ as a particular case of admissibility of pairs of spaces (see Definition 2 below).

It is easy to see that V is a linear operator from d^n to l^n.

The crucial step in the W-transform is the choice of "a reference equation", which has the same shape as the equation to be studied, but already has the desired asymptotic properties

$$x(s+1) = x(s) + \big[(Qx)(s) + g(s)\big]Z(s) \ \ (s \in N_+), \tag{5}$$

where $Q : d^n \to l^n$ is a linear operator and $g \in l^n$.

One usually assumes that for any admissible x_0 there exists a unique (up to the P-equivalence) solution x of the equation (5). In this case, the solution $x_g(s, x_0)$ $(s \in N_+)$ of (5) satisfying $x_g(0, x_0) = x_0$ has the following canonical representation

$$x_g(s, x_0) = U(s)x_0 + (Wg)(s) \ \ (s \in N_+), \tag{6}$$

where $U(s)$ $(s \in N_+)$ is the fundamental matrix to (5) and $W : l^n \to d^n$ is a linear operator such that $(Wg)(0) = 0$ and $(Wg)(s)$ $(s \in N_+)$ is a solution of (5)).

We rewrite the equation (4) using the representation (6) for the reference equation (5) as follows

$$x(s+1) = x(s) + \big[(Qx)(s) + ((V-Q)x)(s) + f(s)\big]Z(s) \ \ (s \in N_+)$$

or alternatively,

$$x(s+1) = x(s) + U(s)x_0 + (W(V-Q)x)(s) + (Wf)(s) \ \ (s \in N_+).$$

Introducing the notation $W(V-Q) = \Theta$, we obtain the equation

$$((I - \Theta)x)(s) = U(s)x_0 + (Wf)(s) \ \ (s \in N_+).$$

To study asymptotic properties of a stochastic difference equation we need a notion of admissibility of a pair of spaces. In the sequel we will use the following spaces of random variables.

The space k^n consists of all n-dimensional $\mathcal{F}_0$-measurable random variables and

$$k_p^n = \Big\{ \alpha : \ \alpha \in k^n, \ \|\alpha\|_{k^n} \overset{def}{=} (E|\alpha|^p)^{1/p} < \infty \Big\}.$$

Given a sequence $\gamma(s)$ $(s \in N_+)$ of positive real numbers, we define two more spaces of discrete stochastic processes:

$$m_p^\gamma = \Big\{ x : \ x \in d^n, \|x\|_{m_p^\gamma} \overset{def}{=} \sup_{s \in N_+} \big(E|\gamma(s)x(s)|^p\big)^{1/p} < \infty \Big\} \ \ (m_p^1 = m_p);$$

and

$$b^\gamma = \big\{f:\ f \in b,\ \gamma f \in b\big\}$$

which is endowed with the induced norm $\|f\|_{b^\gamma} = \|\gamma f\|_b$, where b is a linear subspace of the space l^n equipped with some norm $\|\cdot\|_b$.

Definition 2. We say that the pair (m_p^γ, b^γ) is admissible for the system (4) if there exists a number $\overline{c} \in R_+^1$ such that for any $x_0 \in k_p^n$, $f \in b^\gamma$ we have that $x_f(\cdot, x_0) \in m_p^\gamma$ and

$$\big\|x_f(\cdot, x_0)\big\|_{m_p^\gamma} \le \overline{c}\big(\|x_0\|_{k_p^n} + \|f\|_{b^\gamma}\big). \tag{7}$$

Now we make assumptions on the space b. Letting

$$f = \Big(\sum_{j=-\infty}^{-1} A_1^2(\cdot, j)\varphi(j), \dots, \sum_{j=-\infty}^{-1} A_m^2(\cdot, j)\varphi(j)\Big), \tag{8}$$

we assume that the coefficients of the system (3) satisfy the following condition:

- for any φ such that $\sup\limits_{j<0} E|\varphi(j)|^p < \infty$ the stochastic process (8) belongs to the linear subspace b of the space l^n, the norm in b satisfies the estimate

  $$\|f\|_b \le K \sup_{j<0} \big(E|\varphi(j)|^p\big)^{1/p}$$

  for some positive constant K.

The proofs of the above theorems are based on the following lemmas.

Lemma 1. *Let the pair (m_p^γ, b^γ) be admissible for the reference equation* (5) *and the operator Θ act in the space m_p^γ. If the operator $(I - \Theta_l) : m_p^\gamma \to m_p^\gamma$ is continuously invertible, then the pair (m_p^γ, b^γ) is admissible for the system* (4).

Lemma 2. *If for the system* (4) *corresponding to the equation* (3) *the pair (m_p, b) is admissible, then the trivial solution of* (3) *is p-stable with respect to the initial data.*

Lemma 3. *If for the system* (4) *corresponding to the equation* (3) *the pair (m_p^γ, b^γ) is admissible for some sequence of numbers $\gamma(s)$ $(s \in N_+)$ satisfying $\gamma(s) \ge \delta > 0$ for all $s \in N_+$ $(\delta > 0)$, $\lim\limits_{s\to+\infty} \gamma(s) = +\infty$, then the trivial solution of* (3) *is asymptotically p-stable with respect to the initial data.*

For the technical details of the proofs see the paper [5].

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