Periodic Problem for the Nonlinear Telegraph Equation

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In the plane of independent variables x and t in the strip $\Omega := \{(x,t) \in \mathbb{R}^2 : \ 0 < x < l, \ t \in \mathbb{R}\}$ consider the problem on finding a solution $U(x,t)$ to the nonlinear telegraph equation of the form

$$LU := U_{tt} - U_{xx} + 2aU_t + cU + g(U) = F(x,t), \ \ (x,t) \in \Omega, \tag{1}$$

satisfying the Poincare homogeneous boundary conditions

$$\gamma_1 U_x(0,t) + \gamma_2 U_t(0,t) + \gamma_3 U(0,t) = 0, \ \ t \in \mathbb{R}, \tag{2}$$

and the Dirichlet boundary condition

$$U(l,t) = 0, \ \ t \in \mathbb{R}, \tag{3}$$

respectively, for $x = 0$ and $x = l$, and also the condition of periodicity with respect to the variable t

$$U(x, t+T) = U(x,t), \ \ x \in [0,l], \ \ t \in \mathbb{R}, \tag{4}$$

with constant real coefficients a, c, γ_i, $i = 1,2,3$, with $\gamma_1\gamma_2 \neq 0$. Here $T := const > 0$; F is a given, while U is an unknown real T-periodic in time functions; $g : \mathbb{R} \to \mathbb{R}$ is a given continuous real nonlinear function.

Remark 1. Let $\Omega_T := \Omega \cap \{0 < t < T\}$, $f := F\big|_{\overline{\Omega}_T}$. It is easy to see that if $U \in C^2(\overline{\Omega})$ is a classical solution of the problem (1)–(4), then the function $u := U\big|_{\overline{\Omega}_T}$ is a classical solution of the following nonlocal problem

$$Lu = f(x,t), \ \ (x,t) \in \Omega_T, \tag{5}$$

$$\gamma_1 u_x(0,t) + \gamma_2 u_t(0,t) + \gamma_3 u(0,t) = 0, \ \ u(l,t) = 0, \ \ 0 \leq t \leq T, \tag{6}$$

$$(B_0 u)(x) = 0, \quad (B_0 u_t)(x) = 0, \ \ x \in [0,l], \tag{7}$$

where $(B_\beta w)(x) := w(x,0) - \exp(-\beta T) w(x,T)$, $\beta \in \mathbb{R}$, $x \in [0,l]$, and, vice versa, if $f \in C(\overline{\Omega}_T)$ and $u \in C^2(\overline{\Omega}_T)$ is a classical solution of the problem (5)–(7), then the function $U \in C^2(\overline{\Omega})$, being T-periodic continuation of the function u from the domain Ω_T into the strip Ω, will be a classic solution of the problem (1)–(4), if $f(x,0) = f(x,T)$, $x \in [0,l]$.

Definition 1. Let $f \in C(\overline{\Omega}_T)$ be a given function, and $\Gamma_1 : x = 0, \ 0 \leq t \leq T$, $\Gamma_2 : x = l$, $0 \leq t \leq T$. The function u is called a strong generalized solution of the problem (5)–(7) of the class C, if $u \in C(\overline{\Omega}_T)$ and there exists the sequence of functions $u_n \in \overset{\circ}{C}{}^2(\overline{\Omega}_T, \Gamma_1, \Gamma_2) := \big\{ w \in C^2(\overline{\Omega}_T) : (\gamma_1 w_x + \gamma_2 w_t + \gamma_3 w)|_{\Gamma_1} = 0, \ w|_{\Gamma_2} = 0 \big\}$ such that $u_n \to u$ and $Lu_n \to f$ in the space $C(\overline{\Omega}_T)$, while $B_0 u_n \to 0$ and $B_0 u_{nt} \to 0$ as $n \to \infty$ in the spaces $C^1([0,l])$ and $C([0,l])$, respectively.

Remark 2. It is obvious that a classical solution of the problem (5)–(7) from the space $C^2(\overline{\Omega}_T)$ is a strong generalized solution of this problem of the class C.

Consider the following conditions

$$G(s) := \int\limits_0^s g(s_1)\,ds_1 \geq 0, \quad sg(s) - 2G(s) \geq 0, \quad s \in \mathbb{R}, \tag{8}$$

$$a > 0, \quad c \geq a^2, \quad \gamma_1\gamma_2 < 0, \quad \gamma_3\gamma_2^{-1} \geq a. \tag{9}$$

The following Theorem is valid.

Theorem. *Let $T = 2l$, the conditions* (8), (9) *and $f \in C(\overline{\Omega}_{2l})$ be fulfilled. Then the problem* (5)–(7) *has at least one strong generalized solution u of the class C in the sense of Definition 1, which belongs to the space $C^1(\overline{\Omega}_{2l})$, besides, in the case $f \in C^1(\overline{\Omega}_{2l})$ this solution is a classical one.*

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