Investigation and Approximate Resolution of One Nonlinear Integro-Differential Parabolic Equation

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One type integro-differential nonlinear parabolic model is obtained at mathematical simulation of processes of electromagnetic field penetration in the substance. Based on Maxwell system [1] this model at first appeared in [2]. Many other processes are described by integro-differential system obtained in [2] (see, for example, [3] and references therein). A lot of scientific works are dedicated to investigation and numerical resolution of the initial-boundary value problems for these type models (see, for example, [3] and references therein). The existence, uniqueness and asymptotic behavior of the solution for such type equations and systems are studied in the works [2–6] and in a number of other works as well (for more detail citations see, for example, [3] and references therein).

The present work is dedicated to the investigation and approximate resolution of the initial-boundary value problem with first type boundary conditions for one generalization and one-dimensional variant of such model.

In the domain $Q = (0,1) \times (0,T)$, where T is a positive constant, uniqueness and existence properties and semi-discrete and finite difference approximations are discussed for the following nonlinear integro-differential problem:

$$\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left\{\left[1 + \int_0^t \Big(\frac{\partial U}{\partial x}\Big)^2 d\tau\right]^p \Big|\frac{\partial U}{\partial x}\Big|^{q-2}\frac{\partial U}{\partial x}\right\} = f(x,t), \tag{1}$$

$$U(0,t) = U(1,t) = 0, \tag{2}$$

$$U(x,0) = U_0(x), \tag{3}$$

where p, q are constants and $f = f(x,t)$ and $U_0 = U_0(x)$ are given functions of their arguments.

Principal characteristic peculiarity of the equation (1) is connected with the appearance in the coefficient with derivative of higher order nonlinear term depended on the integral in time. These circumstances requires different discussions than it is usually necessary for the solution of local differential problems.

Using one modification of compactness method developed in [7] (see also [8]) the following existence statement takes place [5].

Theorem 1. *If $0 < p \le 1$, $q \ge 2$, $f \in W_2^1(Q)$, $f(x,0) = 0$, $U_0 \in \overset{\circ}{W}{}_2^1(0,1)$, then there exists the unique solution U of problem (1)–(3) satisfying the following properties:*

$$U \in L_{pq+q}\big(0,T;\overset{\circ}{W}{}^1_{pq+q}(0,1)\big), \quad \frac{\partial U}{\partial t} \in L_2(Q),$$

$$\frac{\partial}{\partial x}\Big(\Big|\frac{\partial U}{\partial x}\Big|^{\frac{q-2}{2}}\frac{\partial U}{\partial x}\Big) \in L_2(Q), \quad \sqrt{T-t}\,\frac{\partial}{\partial t}\Big(\Big|\frac{\partial U}{\partial x}\Big|^{\frac{q-2}{2}}\frac{\partial U}{\partial x}\Big) \in L_2(Q).$$

Here usual well-known spaces are used. The proof of the formulated theorem is divided into several steps applying Galerkin's method and the method of compactness. One of the basic step is to obtain necessary a priori estimates.

In [4], there is proposed the operational scheme with so called conditionally closed operators. That scheme is applied for investigation of problems of (1) types in this work, too.

In order to describe the space-discretization to problem (1)–(3), it is introduced a net whose mesh points are denoted by $x_i = ih$, $i = 0, 1, \dots, M$, with $h = 1/M$. The boundaries are specified by $i = 0$ and $i = M$. The semi-discrete approximation at (x_i, t) is designed $u_i = u_i(t)$. The exact solution to the problem at (x_i, t), denoted by $U_i = U_i(t)$, is assumed to exist and be smooth enough. From the boundary conditions (2) we have $u_0(t) = u_M(t) = 0$. At other points $i = 1, 2, \dots, M-1$, the integro-differential equation will be replaced by approximating the space derivatives by a forward and backward differences. We will use the following known notations.

$$u_{x,i}(t) = \frac{u_{i+1}(t) - u_i(t)}{h}, \quad u_{\overline{x},i}(t) = \frac{u_i(t) - u_{i-1}(t)}{h}.$$

Let's correspond to problem (1)–(3) the following semi-discrete scheme:

$$\frac{du_i}{dt} - \left\{ \left[1 + \int\limits_0^t \left(u_{\overline{x},i} \right)^2 d\tau \right]^p |u_{\overline{x},i}|^{q-2} u_{\overline{x},i} \right\}_{x,i} = f(x_i, t), \quad i = 1, 2, \dots, M-1, \tag{4}$$

$$u_0(t) = u_M(t) = 0, \tag{5}$$

$$u_i(0) = U_{0,i}, \quad i = 0, 1, \dots, M. \tag{6}$$

The (4)–(6) is a Cauchy problem for nonlinear system of ordinary integro-differential equations. Using multiplier $u(t) = (u_1(t), u_2(t), \dots, u_{M-1}(t))$, after simple transformations we obtain the inequality

$$\|u(t)\|^2 + \int\limits_0^t \|u_{\overline{x}}]|^q \, d\tau < C, \tag{7}$$

where C is a positive constant which do not depend on h and norms are defined as follows:

$$(u, v) = \sum_{i=1}^{M-1} u_i v_i h, \quad (u, v] = \sum_{i=1}^{M} u_i v_i h,$$

$$\|u\| = (u, u)^{1/2}, \quad \|u]| = (u, u]^{1/2}.$$

The a priori estimate (7) guarantees the global solvability of problem (4)-(6). It is not difficult to prove the uniqueness of the solution of problem (4)–(6), too.

The following statement takes place.

Theorem 2. *If $0 < p \le 1$, $q \ge 2$, and problem (1)–(3) has a sufficiently smooth solution $U = U(x,t)$, then the solution $u = u(t) = (u_1(t), u_2(t), \dots, u_{M-1}(t))$ of problem (4)–(6) tends to $U = U(t) = (U_1(t), U_2(t), \dots, U_{M-1}(t))$ as $h \to 0$ and the following estimate is true*

$$\|u(t) - U(t)\| < Ch.$$

In order to describe the finite difference method it is introduced a net whose mesh points are denoted by $(x_i, t_j) = (ih, j\tau)$, where $i = 0, 1, \dots, M$; $j = 0, 1, \dots, N$ with $h = \frac{1}{M}$, $\tau = \frac{T}{N}$. The initial line is denoted by $j = 0$. The discrete approximation at (x_i, t_j) is designed by u_i^j once again and the exact solution to problem (1)–(3) by U_i^j.

Using forward derivative formula for time variable and rectangle formula for integration, let us correspond to problem (1)–(3) the following finite difference scheme:

$$\frac{u_i^{j+1}-u_i^j}{\tau}-\left\{\left[1+\tau\sum_{k=1}^{j+1}\left(u_{\overline{x},i}^k\right)^2\right]^p|u_{\overline{x},i}^{j+1}|^{q-2}u_{\overline{x},i}^{j+1}\right\}_{x,i}=f_i^j, \tag{8}$$

$$i=1,2,\dots,M-1;\ \ j=0,1,\dots,N-1,$$

$$u_0^j=u_M^j=0,\ \ j=0,1,\dots,N, \tag{9}$$

$$u_i^0=U_{0,i},\ \ i=0,1,\dots,M. \tag{10}$$

So, system of nonlinear algebraic equations (8)–(10) is obtained. It is not difficult to get the inequality

$$\|u^n\|^2+\sum_{j=1}^n\|u_{\overline{x}}^j]|^q\tau<C,\ \ n=1,2,\dots,N, \tag{11}$$

where C here and below is a positive constant independent of τ and h.

The a priori estimate (11) guarantees the stability of the scheme (8)–(10). Note that it is easy to prove the existence and uniqueness of a solution of the scheme (8)–(10), too.

The following statement takes place.

Theorem 3. *If $p=1$, $q\geq 2$, and problem* (1)–(3) *has a sufficiently smooth solution $U=U(x,t)$, then the solution $u^j=(u_1^j,u_2^j,\dots,u_{M-1}^j)$, $j=1,2,\dots,N$ of difference scheme* (8)–(10) *tends to the $U^j=(U_1^j,U_2^j,\dots,U_{M-1}^j)$, $j=1,2,\dots,N$ as $\tau\to 0$, $h\to 0$ and the following estimate is true*

$$\|u^j-U^j\|<C(\tau+h),\ \ j=1,2,\dots,N.$$

Note that for solving the difference scheme (8)–(10) Newton's iterative process is used. Various numerical experiments are done. These experiments agree with theoretical research.

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