On Limit Irreducibility Sets of Linear Differential Systems

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We consider the linear systems of the form

$$\dot{x} = A(t)x, \;\; x \in \mathbb{R}^n, \;\; t \in I = [0, +\infty), \tag{1_A}$$

with piecewise continuous bounded coefficients ($\|A(t)\| \le a$ for $t \in I$). Along with original systems (1) we will consider perturbed systems (1_{A+Q}) with piecewise continuous perturbations Q defined on I and satisfying either the condition

$$\|Q(t)\| \le C_Q e^{-\sigma t}, \;\; \sigma > 0, \;\; t \ge 0, \tag{2}$$

or the more general condition

$$\lambda[Q] \equiv \overline{\lim_{t\to+\infty}}\, t^{-1} \ln \|Q(t)\| \le -\sigma < 0. \tag{3}$$

If $\sigma = 0$ in (2), (3), then we additionally suppose that $Q(t) \to 0$ as $t \to +\infty$.

Following Yu. S. Bogdanov [1], we say that systems (1_A) and (1_{A+Q}) are asymptotically equivalent (Lyapunov's equivalent, reducible) if there exists a Lyapunov transformation

$$x = L(t)y, \quad \max\Big\{\sup_{t\in I} \|L(t)\|, \; \sup_{t\in I} \|L^{-1}(t)\|, \; \sup_{t\in I} \|\dot{L}(t)\|\Big\} < +\infty,$$

reducing one of them to the other.

The sets $N_2(a,\sigma)$, $N_3(a,\sigma)$, $a \ge 0$, $\sigma \ge 0$, are said to be the irreducibility sets if they consist of all systems (1_A) with the following properties [2]:

1) the norm of the coefficient matrix A is less than or equal to a on I;
2) for each system $(1_A) \in N_i(a,\sigma)$, $i = 2, 3$, there exists a system (1_{A+Q}) with the matrix Q satisfying either the condition (2) or the more general condition (3), respectively, which cannot be reduced to system (1_A).

If Q satisfies (2) or (3) with $\sigma > 2a$, then $\|\int_t^{+\infty} Q(u)\,du\| \le Ce^{-\sigma_1 t}$ for some $C > 0$ and $\sigma_1 > 2a$, therefore [3,5] systems (1_A) and (1_{A+Q}) are asymptotically equivalent, and, therefore, the sets $N_2(a,\sigma)$, $N_3(a,\sigma)$ are empty for all $\sigma > 2a$.

We have [6] the following

Theorem 1. *The following strict inclusions are valid for the irreducibility sets $N_2(a,\sigma)$ and $N_3(a,\sigma)$:*

$$N_i(a_1,\sigma) \subset N_i(a_2,\sigma) \;\; \forall\, 0 \le a_1 < a_2, \;\; \forall\, \sigma \in [0, 2a_2], \;\; i = 2, 3.$$

The limit irreducibility sets

$$N_i(\sigma) \equiv \operatorname*{Lim}_{a\to+\infty} N_i(a,\sigma), \ \ i=2,3,$$

were defined in [4]. The properties of these sets treated as functions of the parameter σ are similar to the properties of the irreducibility sets $N_i(a,\sigma)$, $i=2,3$. By Theorem 1, the limit irreducibility sets are defined as the union of appropriate irreducibility sets

$$\operatorname*{Lim}_{a\to+\infty} N_i(a,\sigma) = \bigcup_{a\geq 0} N_i(a,\sigma),$$

and, by virtue of their definition, they are related by the inclusions $N_2(\sigma) \subseteq N_3(\sigma)$ for all $\sigma \geq 0$. The following statements are valid [6].

Theorem 2. *The limit irreducibility sets $N_2(\sigma)$ and $N_3(\sigma)$ coincide for $\sigma=0$ and do not coincide for any $\sigma>0$, i.e., $N_3(\sigma)\setminus N_2(\sigma)\neq\varnothing$ for any $\sigma>0$.*

Theorem 3. *The limit irreducibility sets $N_2(\sigma)$ and $N_3(\sigma)$ of linear differential n-dimensional systems (1_A) satisfy the strict inclusions*

$$N_i(\sigma_2) \subset N_i(\sigma_1) \ \ \forall\, 0\leq\sigma_1<\sigma_2, \ \ i=2,3.$$

Theorem 4. *The limit irreducibility sets satisfy the relations*

$$\operatorname*{Lim}_{\sigma\to\sigma_0+0} N_i(\sigma) \subset N_i(\sigma_0) \ \ \forall\,\sigma_0\geq 0, \ \ i=2,3,$$
$$\operatorname*{Lim}_{\sigma\to\sigma_0-0} N_2(\sigma) \supset N_2(\sigma_0) \ \ \forall\,\sigma_0>0,$$
$$\operatorname*{Lim}_{\sigma\to\sigma_0-0} N_3(a,\sigma) = N_3(a,\sigma_0) \ \ \forall\,\sigma_0>0.$$

Theorem 5. *The limit sets $N_2(\sigma)$ and $N_3(\sigma)$ are invariant under Lyapunov transformations.*

References

[1] Ju. S. Bogdanov, On asymptotically equivalent linear differential systems. (Russian) *Differencial'nye Uravnenija* **1** (1965), No. 6, 707–716; translation in *Differ. Equations* **1** (1965), No. 6, 541–549.

[2] N. A. Izobov and S. A. Mazanik, A general test for the reducibility of linear differential systems, and the properties of the reducibility coefficient. (Russian) *Differ. Uravn.* **43** (2007), No. 2, 191–202, 286; translation in *Differ. Equ.* **43** (2007), No. 2, 196–207

[3] N. A. Izobov and S. A. Mazanik, On asymptotically equivalent linear systems under exponentially decaying perturbations. (Russian) *Differ. Uravn.* **42** (2006), No. 2, 168–173, 285; translation in *Differ. Equ.* **42** (2006), No. 2, 182–187.

[4] N. A. Izobov and S. A. Mazanik, On sets of linear differential systems to which perturbed linear systems cannot be reduced. (Russian) *Differ. Uravn.* **47** (2011), No. 11, 1545–1550; translation in *Differ. Equ.* **47** (2011), No. 11, 1563–1568

[5] N. A. Izobov and S. A. Mazanik, The coefficient of reducibility of linear differential systems. *Mem. Differential Equations Math. Phys.* **39** (2006), 154–157.

[6] N. A. Izobov and S. A. Mazanik, Parametric properties of irreducibility sets of linear differential systems. (Russian) *Differ. Uravn.* 51 (2015), No. 8, 973–988; trakslation in *Differ. Equ.* **51** (2015), No. 8, 1-11.