The Infinite Version of Perron's Effect of Value Change in Characteristic Exponents in the Neighbourhood of Integer Points

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Just as in our previous report [1], we consider here both the linear differential systems

$$\dot{x} = A(t)x, \ \ x \in R^n, \ \ t \geq t_0 \tag{1}$$

with bounded infinitely differentiable on the semi-axis $[t_0, +\infty)$ coefficients and characteristic exponents $\lambda_1(A) \leq \cdots \leq \lambda_n(A) < 0$, and the nonlinear systems

$$\dot{y} = A(t)y + f(t,y), \ \ y \in R^n, \ \ t \geq t_0 \tag{2}$$

with infinitely differentiable in time t and variables $y_1, \dots, y_n$ so-called m-perturbations $f(t,y)$. These perturbations have the order $m > 1$ of smallness in the neighbourhood of the origin and admissible growth outside of it, satisfying the inequality

$$\|f(t,y)\| \leq C_f \|y\|^m, \quad C_f = const > 0, \ \ y \in R^n, \ \ t \geq t_0. \tag{3}$$

The well-known (partial) Perron's effects of sign and value changes [1], [2, pp. 50–61] in characteristic exponents claimed the existence of such two-dimensional system (1) with specific characteristic exponents $\lambda_1(A) = \lambda_1 < \lambda_2(A) = \lambda_2 < 0$ and the 2-perturbation (3) $f(t,y)$ that all solutions $y(t,c)$, $c \in R^2$ of the two-dimensional perturbed system (2) turned out to be infinitely extendable to the right and had characteristic exponents

$$\lambda\big[y(\cdot,c)\big] = \begin{cases} \lambda_2 < 0, & c = (0,c_2) \neq 0, \\ \lambda_2 > 0, & c_1 \neq 0. \end{cases}$$

The equal to λ_2 coincidence of characteristic exponents of solutions $x(t,c)$ and $y(t,c)$, $c = (c_1,c_2)$ of systems (1) and (2), respectively, on the axis $c_1 = 0$ (for $c_2 \neq 0$) of the plane R^2 as well as the lack of arbitrariness in the parameters $\lambda_1 \leq \lambda_2 < 0$, $m > 1$, and in the set $\beta = \{\lambda[y(\cdot,c)] : \ 0 \neq c \in R^2\}$ just right stipulates its partiality.

To the construction of various complete analogues of Perron's effect of value change in characteristic exponents of differential systems is devoted a cycle of our works, including those written jointly with S. K. Korovin. In particular, in our previous report, for arbitrary parameters $m > 1$, $\lambda_1 \leq \lambda_2 < 0$ and for bounded closed from the above countable set

$$\beta \subset [\lambda_1, +\infty), \ \ \lambda_2 \leq \sup \beta \in \beta,$$

we have stated that there exist the two-dimensional linear system (1) with exponents $\lambda_1(A) = \lambda_1 \leq \lambda_2 = \lambda_2(A)$ and the nonlinear system (2) with m-perturbation (3) such that all its nontrivial solutions $y(t,c)$, $c \in R^2$, are infinitely extendable to the right, and their characteristic exponents form the set $\Lambda(A,f) = \beta$ which coincides for $p = 0 \in R^2$ with its limiting subset

$$\Lambda_p(A,f) \equiv \operatorname*{Lim}_{r \to +0} \Big\{ \lambda\big[y(\cdot,c)\big] : \ 0 < \|c - p\| \leq r \Big\}, \ \ p \in R^2,$$

of characteristic exponents of nontrivial solutions of system (2) starting in any arbitrarily small neighbourhood of the point $p \in R^2$.

In this connection, there arises the problem on the existence of another, different from the origin $(0,0)$, points $p \in R^2$ of the space of initial solutions for which the equality

$$\Lambda(A,f) = \Lambda_p(A,f) = \beta \tag{4}$$

would be fulfilled for an infinite number of points $p = (p_1, p_2) \in R^2$ and for any bounded countable (not necessarily closed from the above) set β of positive, in particular, numbers. Its solution is involved in the following theorem.

Theorem. *For any parameters $m > 1$, $\lambda_1 \leq \lambda_2 < 0$ and for any finite or bounded countable set*

$$\beta \subset [\lambda_1, +\infty), \quad \beta \cap [\lambda_2, +\infty) \neq \varnothing,$$

there exist:

1) *the two-dimensional system* (1) *with characteristic exponents $\lambda_1(A) = \lambda_1 \leq \lambda_2 = \lambda_2(A)$;*
2) *the infinitely differentiable with respect to the variables t, y_1, y_2, and satisfying the condition* (3) *perturbation $f : [1,+\infty) \times R^2 \to R^2$ of order $m > 1$ such that all nontrivial solutions of the nonlinear two-dimensional system* (2) *with linear approximation* (1) *are infinitely extendable to the right, and their characteristic exponents form the set $\Lambda(A,f) = \beta$ which takes at the points $p = (p_1, p_2) \in R^2$ with integer coordinates its limiting values*

$$\Lambda_p(A,f) = \begin{cases} \beta & \text{if } p_1 \in Z, \ \ p_2 = 0, \\ \beta \cap [\lambda_2, +\infty) & \text{if } p_1 \in Z, \ \ p_2 \in Z \setminus \{0\}. \end{cases} \tag{5}$$

Statement (5) and condition (4) result in the following

Corollary 1. *In the case of a finite or bounded countable set $\beta \subset (0, +\infty)$ of positive numbers, the representation*

$$\Lambda(A,f) = \Lambda_p(A,f), \ \ p_1 \in Z, \ \ p_2 \in Z$$

is valid.

When proving the theorem in the case, where

$$\beta \cap [\lambda_2, +\infty) \neq \beta,$$

we have obtained a stronger compared with the second statement in (5)

Corollary 2. *For the limiting at the point $p = (p_1, p_2) \in R^2$ set $\Lambda_p(A,f)$ of characteristic exponents of solutions of the perturbed system* (2)*, the representation*

$$\Lambda_p(A,f) = \beta \cap [\lambda_2 + \infty) \neq \beta, \ \ p_1 \in R, \ \ p_2 \in Z \setminus \{0\}$$

is valid.

The results obtained in the present report are published in [1]– [3].

References

[1] A. V. Il'in and N. A. Izobov, The infinite analogues of Perrons effect of value change in characteristic exponents. *Abstracts of the International Workshop on the Qualitative Theory of Differential Equations – QUALITDE-2014, Tbilisi, Georgia, December 18-20, 2014*, pp. 51–52; http://www.rmi.ge/eng/QUALITDE-2014/workshop_2014.htm.

[2] A. V. Il'in and N. A. Izobov, Countable analogue of Perron's effect of value change in characteristic exponents in any neighbourhood of the origin. (Russian) *Differentsial'nye Uravneniya* **51** (2015), No. 8, 1115–1117.

[3] N. A. Izobov and A. V. Il'in, The Perron's effect of infinite value change in characteristic exponents in any neighbourhood of the origin. (Russian) *Differentsial'nye Uravneniya* **51** (2015), No. 11, 1420–1432.