Asymptotic Representations of Solutions of Second-Order Differential Equations with Rapidly Varying Nonlinearities

V. M. Evtukhov, A. G. Chernikova

Odessa I. I. Mechnikov National University, Odessa, Ukraine
E-mail: evmod@i.ua; anastacia.chernikova@gmail.com

We consider the differential equation

$$y'' = \alpha_0 p(t)\varphi(y) \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\to]0, +\infty[$ is a continuous function, $\varphi : \Delta_{Y_0} \to]0; +\infty[$ $(i = \overline{1, n})$ is a continuously differentiable function satisfying the conditions

$$\varphi'(y) \neq 0 \ \text{ at } \ y \in \Delta_{Y_0}, \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \varphi(y) = \begin{cases} \text{either} & 0, \\ \text{or} & +\infty, \end{cases} \quad \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{\varphi''(y)\varphi(y)}{\varphi'^2(y)} = 1, \tag{2}$$

where Δ_{Y_0} is some one-sided neighborhood of the points Y_0, Y_0 is equal to either 0 or $\pm\infty$.

From the identity

$$\frac{\varphi''(y)\varphi(y)}{\varphi'^2(y)} = \frac{\big(\frac{\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{\varphi'(y)}{\varphi(y)}\big)^2} + 1$$

and the conditions (2) it follows that

$$\frac{\varphi'(y)}{\varphi(y)} \sim \frac{\varphi''(y)}{\varphi'(y)} \ \text{ as } \ y \to Y_0 \ \ (y \in \Delta_{Y_0}) \ \text{ and } \ \lim_{\substack{y \to Y_0 \\ y \in \Delta_{Y_0}}} \frac{y\varphi'(y)}{\varphi(y)} = \pm\infty.$$

The function φ in the equation (1) and its derivative of the first order are (see, Seneta E. [1, Ch. 3, § 3.4, pp. 91–92]) rapidly varying as $y \to Y_0$.

The most simple example of such a function is the function $\varphi(y) = e^{\sigma y}$ $(\sigma \neq 0)$ as $Y_0 = +\infty$. In case of such function φ the asymptotic behaviour of solutions of the differential equation (1) was studied in [2–6].

Under conditions (2) in the monography by V. Maric [7, Ch. 3, § 3, pp. 90–99] for the case when $\alpha_0 = 1$, $\omega = +\infty$, $Y_0 = 0$ and p-regularly varying function as $t \to +\infty$, and in [8] for the general case, asymptotic representations for some classes of solutions of the differential equation (1) have been established. Thus in [8] a class of studied solutions was defined through the function φ.

Naturally, however, it is represented for the equation (1) to investigate the same class of solutions, which was studied earlier (see, for example, [9]) in case of regularly varying as $y \to Y_0$ nonlinearity φ.

Definition. A solution y of the equation (1) is called a $P_\omega(Y_0, \lambda_0)$-solution, where $-\infty \leq \lambda_0 \leq +\infty$, if it is defined on some interval $[t_0, \omega[\subset [a, \omega[$ and satisfies the following conditions:

$$\lim_{t \uparrow \omega} y(t) = Y_0, \quad \lim_{t \uparrow \omega} y'(t) = \begin{cases} \text{either} & 0, \\ \text{or} & \pm\infty, \end{cases} \quad \lim_{t \uparrow \omega} \frac{y'^2(t)}{y''(t)y(t)} = \lambda_0.$$

The aim of the paper is to derive necessary and sufficient conditions for the existence of $P_\omega(\Lambda_0)$-solutions of the equation (1) when $\lambda_0 \in \mathbb{R}\setminus\{0;1\}$, and also to establish asymptotic formulas for such solutions and their derivatives of the first order.

Let

$$\Delta_{Y_0}=\begin{cases}[y_0,Y_0[\,, & \text{if } \Delta_{Y_0} \text{ is a left neighborhood of } Y_0,\\ ]Y_0,y_0], & \text{if } \Delta_{Y_0} \text{ is a right neighborhood of } Y_0,\end{cases}$$

where $|y_0|<1$, if $Y_0=0$, and $y_0>1$ ($y_0<-1$), if $Y_0=+\infty$ ($Y_0=-\infty$).

We set

$$\nu_0=\operatorname{sign} y_0, \quad \mu_0=\operatorname{sign}\varphi'(y),$$

$$\pi_\omega(t)=\begin{cases}t, & \text{if } \omega=+\infty,\\ t-\omega, & \text{if } \omega<+\infty,\end{cases} \quad J(t)=\int\limits_A^t \pi_\omega(\tau)p(\tau)\,d\tau, \quad \Phi(y)=\int\limits_B^y \frac{ds}{\varphi(s)},$$

$$q(t)=\frac{\alpha_0(\lambda_0-1)\pi_\omega^2(t)\varphi(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)))}{\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))}\,,$$

$$H(t)=\frac{\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))\varphi'(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)))}{\varphi(\Phi^{-1}(\alpha_0(\lambda_0-1)J(t)))}\,,$$

where

$$A=\begin{cases}\omega, & \text{if } \int\limits_a^\omega |\pi_\omega(\tau)|p(\tau)\,d\tau<+\infty,\\ a, & \text{if } \int\limits_a^\omega |\pi_\omega(\tau)|p(\tau)\,d\tau=\pm\infty,\end{cases} \qquad B=\begin{cases}Y_0, & \text{if } \int\limits_{y_0}^{Y_0}\frac{ds}{\varphi(s)}=\text{const},\\ y_0, & \text{if } \int\limits_{y_0}^{Y_0}\frac{ds}{\varphi(s)}=\pm\infty.\end{cases}$$

With use of properties of rapidly varying functions (see, Bingham N. H., Goldie C. M., Teugels J. L. [10, Ch. 3, 3.10, pp. 174–178]) and the results from [11] on the existence of systems of quasilinear differential equations with vanishing solutions in singular point, the following two theorems are established.

Theorem 1. *Let $\lambda_0 \in \mathbb{R}\setminus\{0;1\}$. Then for the existence of $P_\omega(\Lambda_0)$-solutions of the equation* (1) *it is necessary that*

$$\alpha_0\nu_0\lambda_0>0, \quad \alpha_0\mu_0(\lambda_0-1)J(t)<0 \ \ \text{at} \ \ t\in\,]a,\omega[\,, \tag{3}$$

$$\lim_{t\uparrow\omega}\frac{\pi_\omega(t)J'(t)}{J(t)}=\pm\infty, \quad \lim_{t\uparrow\omega}q(t)=\frac{\lambda_0}{\lambda_0-1}\,. \tag{4}$$

Moreover, each solution of this kind admits the following asymptotic representation:

$$y(t)=\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))\Big[1+\frac{o(1)}{H(t)}\Big] \ \ \text{at} \ \ t\uparrow\omega, \tag{5}$$

$$y'(t)=\frac{\lambda_0}{\lambda_0-1}\,\frac{\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))}{\pi_\omega(t)}\,[1+o(1)] \ \ \text{at} \ \ t\uparrow\omega. \tag{6}$$

Theorem 2. *Let $\lambda_0 \in \mathbb{R}\setminus\{0;1\}$, conditions* (3), (4) *be satisfied and there exist a final or equal to infinity*

$$\lim_{\substack{y\to Y_0\\ y\in\Delta_{Y_0}}}\frac{(\frac{\varphi'(y)}{\varphi(y)})'}{(\frac{\varphi'(y)}{\varphi(y)})^2}\sqrt{\Big|\frac{y\varphi'(y)}{\varphi(y)}\Big|}\,.$$

Then:

1) *if*

$$(\lambda_0-1)J(t)<0 \ \ at \ \ t\in\,]a,\omega[\ \ and \ \ \lim_{t\uparrow\omega}\Big[\frac{\lambda_0}{\lambda_0-1}-q(t)\Big]|H(t)|^{\frac{1}{2}}=0,$$

the differential equation (1) *has a one-parametric family of* $P_\omega(Y_0,\lambda_0)$*-solutions with asymptotic representations* (5), (6), *and the derivative of such solutions admits the representation*

$$y'(t)=\frac{\lambda_0}{\lambda_0-1}\,\frac{\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))}{\pi_\omega(t)}\,\big[1+|H(t)|^{-\frac{1}{2}}o(1)\big] \ \ at \ \ t\uparrow\omega;$$

2) *if*

$$(\lambda_0-1)J(t)>0 \ \ at \ \ t\in\,]a,\omega[\,, \quad \lim_{t\uparrow\omega}\Big[\frac{\lambda_0}{\lambda_0-1}-q(t)\Big]|H(t)|^{\frac{1}{2}}\bigg(\int\limits_{t_0}^{t}\frac{|H(\tau)|^{\frac{1}{2}}\,d\tau}{\pi_\omega(\tau)}\bigg)^2=0$$

and

$$\lim_{t\uparrow\omega}\frac{\int\limits_{t_0}^{t}\frac{|H(\tau)|^{\frac{1}{2}}\,d\tau}{\pi_\omega(\tau)}}{|H(t)|^{\frac{1}{2}}}=0, \quad \lim_{t\uparrow\omega}|H(t)|^{\frac{1}{2}}\bigg(\int\limits_{t_0}^{t}\frac{|H(\tau)|^{\frac{1}{2}}\,d\tau}{\pi_\omega(\tau)}\bigg)\,\frac{\big(\frac{y\varphi'(y)}{\varphi(y)}\big)'}{\big(\frac{y\varphi'(y)}{\varphi(y)}\big)^2}\bigg|_{y=\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))}=0,$$

where t_0 *– some number from* $[a,\omega[\,$, *the differential equation* (1) *as* $\omega=+\infty$ *has a one-parametric family of* $P_\omega(Y_0,\lambda_0)$*-solutions admitting the asymptotic representations*

$$y(t)=\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))\bigg[1+\bigg(H(t)\int\limits_{t_0}^{t}\frac{|H(\tau)|^{\frac{1}{2}}\,d\tau}{\pi_\omega(\tau)}\bigg)^{-1}o(1)\bigg] \ \ at \ \ t\uparrow\omega,$$

$$y'(t)=\frac{\lambda_0}{\lambda_0-1}\,\frac{\Phi^{-1}(\alpha_0(\lambda_0-1)J(t))}{\pi_\omega(t)}\,\bigg[1+\bigg(\int\limits_{t_0}^{t}\frac{|H(\tau)|^{\frac{1}{2}}\,d\tau}{\pi_\omega(\tau)}\bigg)^{-1}o(1)\bigg] \ \ at \ \ t\uparrow\omega,$$

and for $\omega<+\infty$, *a two-parametric families of* $P_\omega(Y_0,\lambda_0)$*-solutions with such representations.*

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