Asymptotic Representations of Solutions of Second-Order Differential Equations with Rapidly Varying Nonlinearities

V. M. Evtukhov, A. G. Chernikova
Odessa I. I. Mechnikov National University, Odessa, Ukraine
E-mail: evmod@i.ua; anastacia.chernikova@gmail.com

We consider the differential equation

$$y'' = \alpha_0 p(t) \varphi(y)$$

(1)

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega] \to [0, +\infty]$ is a continuous function, $\varphi : \Delta Y_0 \to [0; +\infty]$ $(i = 1, n)$ is a continuously differentiable function satisfying the conditions

$$\varphi'(y) \neq 0 \text{ at } y \in \Delta Y_0, \quad \lim_{y \to Y_0} \varphi(y) = \begin{cases} \text{either } 0, & \lim_{y \to Y_0} \frac{\varphi''(y) \varphi(y)}{\varphi'(y)} = 1, \\ \text{or } +\infty, & \text{if } Y_0 \text{ is equal to either } 0 \text{ or } \pm \infty. \end{cases}$$

(2)

where $\Delta Y_0$ is some one-sided neighborhood of the points $Y_0$. From the identity

$$\frac{\varphi''(y) \varphi(y)}{\varphi'(y)^2} = \left(\frac{\varphi(y)}{\varphi'(y)}\right)^2 + 1$$

and the conditions (2) it follows that

$$\frac{\varphi'(y)}{\varphi(y)} \sim \frac{\varphi''(y)}{\varphi'(y)} \quad \text{as } y \to Y_0 \quad (y \in \Delta Y_0) \quad \text{and} \quad \lim_{y \to Y_0} \frac{y \varphi'(y)}{\varphi(y)} = \pm \infty.$$

The function $\varphi$ in the equation (1) and its derivative of the first order are (see, Seneta E. [1, Ch. 3, § 3.4, pp. 91–92]) rapidly varying as $y \to Y_0$.

The most simple example of such a function is the function $\varphi(y) = e^{\sigma y}$ ($\sigma \neq 0$) as $Y_0 = +\infty$. In case of such function $\varphi$, the asymptotic behaviour of solutions of the differential equation (1) was studied in [2–6].

Under conditions (2) in the monography by V. Maric [7, Ch. 3, § 3, pp. 90–99] for the case when $\alpha_0 = 1$, $\omega = +\infty$, $Y_0 = 0$ and $p$-regularly varying function as $t \to +\infty$, and in [8] for the general case, asymptotic representations for some classes of solutions of the differential equation (1) have been established. Thus in [8] a class of studied solutions was defined through the function $\varphi$.

Naturally, however, it is represented for the equation (1) to investigate the same class of solutions, which was studied earlier (see, for example, [9]) in case of regularly varying as $y \to Y_0$ nonlinearity $\varphi$.

Definition. A solution $y$ of the equation (1) is called a $P_\omega(Y_0, \lambda_0)$-solution, where $-\infty \leq \lambda_0 \leq +\infty$, if it is defined on some interval $[t_0, \omega] \subset [a, \omega]$ and satisfies the following conditions:

$$\lim_{t \uparrow \omega} y(t) = Y_0, \quad \lim_{t \uparrow \omega} y'(t) = \begin{cases} \text{either } 0, & \lim_{t \uparrow \omega} \frac{y'(t)}{y(t)} = \lambda_0, \\ \text{or } \pm \infty, & \lim_{t \uparrow \omega} y(t) = \lambda_0. \end{cases}$$
The aim of the paper is to derive necessary and sufficient conditions for the existence of $P_{\omega}(\Lambda_0)$-solutions of the equation (1) when $\lambda_0 \in \mathbb{R} \setminus \{0; 1\}$, and also to establish asymptotic formulas for such solutions and their derivatives of the first order.

Let

$$\Delta Y_0 = \begin{cases} [y_0, Y_0[, & \text{if } \Delta Y_0 \text{ is a left neighborhood of } Y_0, \\ [Y_0, y_0], & \text{if } \Delta Y_0 \text{ is a right neighborhood of } Y_0, \end{cases}$$

where $|y_0| < 1$, if $Y_0 = 0$, and $y_0 > 1$ ($y_0 < -1$), if $Y_0 = +\infty$ ($Y_0 = -\infty$).

We set

$$\nu_0 = \text{sign } y_0, \quad \mu_0 = \text{sign } \varphi'(y),$$

$$\pi_\omega(t) = \begin{cases} t, & \text{if } \omega = +\infty, \\ t - \omega, & \text{if } \omega < +\infty, \end{cases} \quad J(t) = \int_t^\omega \pi_\omega(\tau)p(\tau) \, d\tau, \quad \Phi(y) = \int_B^y \frac{ds}{\varphi(s)},$$

$$q(t) = \frac{\alpha_0(\lambda_0 - 1)\pi_\omega(t)\varphi(\Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t)))}{\Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t))},$$

$$H(t) = \frac{\Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t))\varphi'(\Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t)))}{\varphi(\Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t)))},$$

where

$$A = \begin{cases} \omega, & \text{if } \int_a^\omega |\pi_\omega(\tau)|p(\tau) \, d\tau < +\infty, \\ a, & \text{if } \int_a^\omega |\pi_\omega(\tau)|p(\tau) \, d\tau = \pm\infty, \end{cases} \quad B = \begin{cases} Y_0, & \text{if } \int_{y_0}^{Y_0} \frac{ds}{\varphi(s)} = \text{const}, \\ y_0, & \text{if } \int_{y_0}^{Y_0} \frac{ds}{\varphi(s)} = \pm\infty. \end{cases}$$

With use of properties of rapidly varying functions (see, Bingham N. H., Goldie C. M., Teugels J. L. [10, Ch. 3, 3.10, pp. 174–178]) and the results from [11] on the existence of systems of quasilinear differential equations with vanishing solutions in singular point, the following two theorems are established.

**Theorem 1.** Let $\lambda_0 \in \mathbb{R} \setminus \{0; 1\}$. Then for the existence of $P_{\omega}(\Lambda_0)$-solutions of the equation (1) it is necessary that

$$\alpha_0\nu_0\lambda_0 > 0, \quad \alpha_0\mu_0(\lambda_0 - 1)J(t) < 0 \quad \text{at } t \in [a, \omega[, \quad \lim_{t \uparrow \omega} \frac{\pi_\omega(t)J'(t)}{J(t)} = \pm\infty, \quad \lim_{t \uparrow \omega} q(t) = \frac{\lambda_0}{\lambda_0 - 1}. \quad \text{(3)}$$

Moreover, each solution of this kind admits the following asymptotic representation:

$$y(t) = \Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t)) \left[1 + \frac{o(1)}{H(t)}\right] \quad \text{at } t \uparrow \omega, \quad \text{(5)}$$

$$y'(t) = \frac{\lambda_0}{\lambda_0 - 1} \frac{\Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t))}{\pi_\omega(t)} [1 + o(1)] \quad \text{at } t \uparrow \omega. \quad \text{(6)}$$

**Theorem 2.** Let $\lambda_0 \in \mathbb{R} \setminus \{0; 1\}$, conditions (3), (4) be satisfied and there exist a final or equal to infinity

$$\lim_{y \to Y_0} \frac{\varphi'(y)}{\varphi(y)} = \sqrt{\frac{y\varphi'(y)}{\varphi(y)}}.$$

Then:
1) if

\[(\lambda_0 - 1)J(t) < 0 \quad \text{at} \quad t \in [a, \omega], \quad \text{and} \quad \lim_{t \uparrow \omega} \left[ \frac{\lambda_0}{\lambda_0 - 1} - q(t) \right] |H(t)|^{1/2} = 0,\]

the differential equation (1) has a one-parametric family of \(P_\omega(Y_0, \lambda_0)\)-solutions with asymptotic representations (5), (6), and the derivative of such solutions admits the representation

\[y'(t) = \frac{\lambda_0}{\lambda_0 - 1} \Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t)) \left[ 1 + \frac{\int^t_0 |H(\tau)|^{1/2} d\tau}{\pi_\omega(\tau)} \right]^{1/2} o(1) \quad \text{at} \quad t \uparrow \omega;\]

2) if

\[(\lambda_0 - 1)J(t) > 0 \quad \text{at} \quad t \in [a, \omega], \quad \lim_{t \uparrow \omega} \left[ \frac{\lambda_0}{\lambda_0 - 1} - q(t) \right] |H(t)|^{1/2} \left( \int^t_0 \frac{|H(\tau)|^{1/2} d\tau}{\pi_\omega(\tau)} \right)^2 = 0\]

and

\[\lim_{t \uparrow \omega} \int^t_0 \frac{|H(\tau)|^{1/2} d\tau}{\pi_\omega(\tau)} = 0, \quad \lim_{t \uparrow \omega} \left| H(t) \right|^{1/2} \left( \int^t_0 \frac{|H(\tau)|^{1/2} d\tau}{\pi_\omega(\tau)} \right) \left| \frac{\phi'(y)}{\phi(y)} \right|^2 \left| \frac{\phi'(y)}{\phi(y)} \right| = 0,\]

where \(t_0\) — some number from \([a, \omega]\), the differential equation (1) as \(\omega = +\infty\) has a one-parametric family of \(P_\omega(Y_0, \lambda_0)\)-solutions admitting the asymptotic representations

\[y(t) = \Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t)) \left[ 1 + \left( \int^t_{t_0} \frac{|H(\tau)|^{1/2} d\tau}{\pi_\omega(\tau)} \right)^{-1} o(1) \right] \quad \text{at} \quad t \uparrow \omega,\]

\[y'(t) = \frac{\lambda_0}{\lambda_0 - 1} \Phi^{-1}(\alpha_0(\lambda_0 - 1)J(t)) \left[ 1 + \left( \int^t_{t_0} \frac{|H(\tau)|^{1/2} d\tau}{\pi_\omega(\tau)} \right)^{-1} o(1) \right] \quad \text{at} \quad t \uparrow \omega,\]

and for \(\omega < +\infty\), a two-parametric families of \(P_\omega(Y_0, \lambda_0)\)-solutions with such representations.

References


