Multi-Point Boundary Value Problems for Functional Differential Equations

Alexander Domoshnitsky

Department of Mathematics and Computer Sciences, Ariel University of Samaria, Israel
E-mail: adom@ariel.ac.il

Robert Hakl

Institute of Mathematics, Czech Academy of Sciences, Brno Branch, Brno, Czech Republic
E-mail: hakl@ipm.cz

Bedřich Půža

Institute of Mathematics, Czech Academy of Sciences, branch in Brno, Brno, Czech Republic;
Institute of Informatics, FBM, Brno University of Technology, Brno, Czech Republic
E-mail: puza@math.cas.cz; puza@fbm.vutbr.cz

On the interval $[a,b]$, we consider the multi-point boundary value problem

$$u'(t)=\ell(u)(t)+q(t), \tag{1}$$

$$\sum_{i=1}^{n}\alpha_i u(t_i)=c, \tag{2}$$

where $\ell : C([a,b];\mathbb{R})\to L([a,b];\mathbb{R})$ is a linear bounded operator, $q\in L([a,b];\mathbb{R})$, $\alpha_i\in\mathbb{R}\setminus\{0\}$, $a\le t_1<t_2<\cdots<t_n\le b$ $(i=1,\dots,n)$, and $c\in\mathbb{R}$. Here and in what follows, $C([a,b];\mathbb{R})$ and $L([a,b];\mathbb{R})$ stand for Banach spaces of continuous and Lebesgue integrable functions defined on $[a,b]$, respectively, with standard norms; $C([a,b];\mathbb{R}_+)$ and $L([a,b];\mathbb{R}_+)$ are subsets of non-negative functions of the corresponding spaces; $AC([a,b];\mathbb{R})$ is a set of absolutely continuous functions defined on $[a,b]$.

A linear bounded operator $\ell : C([a,b];\mathbb{R})\to L([a,b];\mathbb{R})$ is called an a-Volterra operator, resp. a b-Volterra operator, if for arbitrary $c\in\,]a,b]$, resp. $c\in[a,b[\,$, and $v\in C([a,b];\mathbb{R})$ such that

$$v(t)=0 \ \text{ for } \ t\in[a,c], \ \text{ resp. } \ v(t)=0 \ \text{ for } \ t\in[c,b],$$

the equality

$$\ell(v)(t)=0 \ \text{ for a.e. } \ t\in[a,c], \ \text{ resp. } \ \ell(v)(t)=0 \ \text{ for a.e. } \ t\in[c,b],$$

is fulfilled.

Notation. Let $\ell : C([a,b];\mathbb{R})\to L([a,b];\mathbb{R})$ be a linear bounded operator. Then $\ell\in\mathcal{P}_{ab}$ iff it transforms a set $C([a,b];\mathbb{R}_+)$ into a set $L([a,b];\mathbb{R}_+)$; $\ell\in\mathcal{P}_{ab}^+$ iff it transforms the non-negative non-decreasing absolutely continuous functions to the non-negative functions; $\ell\in\mathcal{S}_{ab}(a)$, resp. $\ell\in\mathcal{S}_{ab}(b)$, iff every absolutely continuous function u satisfying

$$u'(t)\ge\ell(u)(t) \ \text{ for a.e. } \ t\in[a,b], \ \ u(a)\ge0,$$

resp.

$$u'(t)\le\ell(u)(t) \ \text{ for a.e. } \ t\in[a,b], \ \ (b)\ge0,$$

admits the inequality $u(t)\ge0$ for $t\in[a,b]$.

Remark 1. In the case when $\ell(u)(t) \stackrel{\text{def}}{=} p(t)u(\tau(t)) - g(t)u(\mu(t))$ with $p, g \in L([a,b];\mathbb{R}_+)$, $\tau, \mu : [a,b] \to [a,b]$ measurables functions, it can be shown that $\ell \in \mathcal{P}^+_{ab}$ iff $p(t) \ge g(t)$ and $g(t)(\tau(t) - \mu(t)) \ge 0$ for a.e. $t \in [a,b]$.

The efficient conditions guaranteeing the inclusions $\ell \in \mathcal{S}_{ab}(a)$ and $\ell \in \mathcal{S}_{ab}(b)$ can be found in [2].

The proofs of the following theorems are based on the results established in [1].

Theorem 1. *Let $\ell \in \mathcal{P}^+_{ab}$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $\ell_0 \in \mathcal{S}_{ab}(a)$. Let, moreover, there exist $i_j \in \{1, \dots, n\}$ $(j = 1, \dots, k)$ such that*

$$n > i_1 > i_2 > \cdots > i_k \ge 1, \tag{3}$$

and either

$$(-1)^r \alpha_z > 0 \ \ for \ \ z = i_{r+1} + 1, \dots, i_r \ \ (r = 0, \dots, k) \tag{4}$$

or

$$(-1)^r \alpha_z < 0 \ \ for \ \ z = i_{r+1} + 1, \dots, i_r \ \ (r = 0, \dots, k), \tag{5}$$

where $i_0 = n$, $i_{k+1} = 0$. Let, in addition,

$$\sum_{z=i_{2r+1}+1}^{i_{2r}} |\alpha_z| \ge \sum_{z=i_{2r+2}+1}^{i_{2r+1}} |\alpha_z|, \ \ r = 0, \dots, \Big[\frac{k-1}{2}\Big]. \tag{6}$$

If either at least one of the inequalities in (6) is strict, or k is even, or

$$\int\limits_I \ell(1)(t)\,dt \neq 0, \quad I = \bigcup_{r=0}^{[\frac{k-1}{2}]} [t_{i_{2r+2}+1}, t_{i_{2r}}], \tag{7}$$

then the problem (1), (2) is uniquely solvable.

Theorem 2. *Let $\ell \in \mathcal{P}^+_{ab}$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$. Let, moreover, there exist $\gamma \in AC([a,b];\mathbb{R})$ satisfying*

$$\gamma(t) > 0 \ \ for \ \ t \in [a,b], \tag{8}$$

$$\gamma'(t) \ge \ell(\gamma)(t) \ \ for \ a.e. \ \ t \in [a,b], \tag{9}$$

and let there exist $i_j \in \{1, \dots, n\}$ $(j = 1, \dots, k)$ such that (3) holds and either (4) or (5) is satisfied, where $i_0 = n$, $i_{k+1} = 0$. Let, in addition, (6) be fulfilled. If either at least one of the inequalities in (6) is strict, or k is even, or (7) holds, then the problem (1), (2) is uniquely solvable.

Theorem 3. *Let $\ell \in \mathcal{P}^+_{ab}$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $\ell_0 \in \mathcal{S}_{ab}(a)$. Let, moreover, there exist $i_j \in \{1, \dots, n\}$ $(j = 1, \dots, k)$ such that (3) holds, and either (4) or (5) be fulfilled where $i_0 = n$, $i_{k+1} = 0$. Let, in addition,*

$$\frac{\gamma(t_n)}{\gamma(a)} \sum_{z=i_1+1}^{n} |\alpha_z| \le \sum_{z=1}^{i_k} |\alpha_z| \ \ if \ k \ is \ odd, \tag{10}$$

$$\frac{\gamma(t_n)}{\gamma(a)} \sum_{z=i_1+1}^{n} |\alpha_z| + \sum_{z=1}^{i_k} |\alpha_z| \le \sum_{z=i_k+1}^{i_{k-1}} |\alpha_z| \ \ if \ k \ is \ even, \tag{11}$$

and

$$\sum_{z=i_{2r+3}+1}^{i_{2r+2}} |\alpha_z| \le \sum_{z=i_{2r+2}+1}^{i_{2r+1}} |\alpha_z|, \ \ r = 0, \dots, \Big[\frac{k-3}{2}\Big] \ \ if \ k \ge 3, \tag{12}$$

where $\gamma \in AC([a,b];\mathbb{R})$ is a function satisfying (8) *and* (9)[1]. *If either at least one of the inequalities in* (10)–(12) *is strict, or there exists $I \subseteq [a,t_n]$ with* $\operatorname{meas} I > 0$ *such that*

$$\gamma'(t) \neq \ell(\gamma)(t) \ \text{ for a.e. } \ t \in I, \tag{13}$$

or

$$\sum_{i=1}^{n} \alpha_i \gamma(t_i) \neq 0, \tag{14}$$

or

$$\int\limits_{I} \ell(1)(t)dt \neq 0, \tag{15}$$

where

$$\begin{gathered} I = [t_{i_1}, t_n] \cup I_1 \cup I_2, \\ I_1 = [a, t_{i_k}] \ \text{ if } k \text{ is odd}, \quad I_1 = [a, t_{i_{k-1}}] \ \text{ if } k \text{ is even}, \\ I_2 = \bigcup_{r=0}^{[\frac{k-3}{2}]} [t_{i_{2r+3}+1}, t_{i_{2r+1}}] \ \text{ if } k \geq 3, \quad I_2 = \varnothing \ \text{ if } k < 3, \end{gathered} \tag{16}$$

then the problem (1), (2) *is uniquely solvable.*

Theorem 4. *Let $\ell \in \mathcal{P}_{ab}^{+}$ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$ and let $-\ell_1 \in \mathcal{S}_{ab}(b)$. Let, moreover, there exist $\gamma \in AC([a,b];\mathbb{R})$ satisfying* (8) *and* (9), *and let there exist $i_j \in \{1,\dots,n\}$ ($j = 1,\dots,k$) such that* (3) *holds, and either* (4) *or* (5) *be fulfilled where $i_0 = n$, $i_{k+1} = 0$. Let, in addition,* (10)–(12) *be satisfied. If either at least one of the inequalities in* (10)–(12) *is strict, or there exists $I \subseteq [a,t_n]$ with* $\operatorname{meas} I > 0$ *such that* (13) *holds, or* (14), *or* (15) *is fulfilled with I defined by* (16), *then the problem* (1), (2) *is uniquely solvable.*

Theorem 5. *Let ℓ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, $\ell(1)(t) \geq 0$ for a.e. $t \in [a,b]$, and let $-\ell_1 \in \mathcal{S}_{ab}(b)$ be an a-Volterra operator. Let, moreover, there exist $\gamma \in AC([a,b];\mathbb{R})$ satisfying* (8) *and* (9). *Let, in addition, $t_1 = a$ and*

$$\alpha_1 \alpha_i < 0 \ \ (i = 2,\dots,n), \quad |\alpha_1| \leq \sum_{i=2}^{n} |\alpha_i|.$$

If either

$$|\alpha_1| < \sum_{i=2}^{n} |\alpha_i|$$

or

$$\int\limits_{a}^{t_n} \ell(1)(t)\,dt \neq 0,$$

then the problem (1), (2) *is uniquely solvable.*

Theorem 6. *Let ℓ admit the representation $\ell = \ell_0 - \ell_1$ with $\ell_0, \ell_1 \in \mathcal{P}_{ab}$, $\ell(1)(t) \geq 0$ for a.e. $t \in [a,b]$, and let $\ell_0 \in \mathcal{S}_{ab}(a)$ be a b-Volterra operator. Let, moreover, $t_n = b$ and*

$$|\alpha_n| \geq \sum_{i=1}^{n-1} \sigma_i |\alpha_i|,$$

where

$$\sigma_i = \frac{1}{2}\big(1 - \operatorname{sgn}(\alpha_i \alpha_n)\big) \ \ (i = 1,\dots,n-1).$$

Let, in addition, at least one of the following items be fulfilled:

[1]The existence of such a function is guaranteed by [2, Theorem 1.1].

(a)
$$|\alpha_n| > \sum_{i=1}^{n-1} \sigma_i |\alpha_i|;$$

(b) *there exists* $i_0 \in \{1, \dots, n-1\}$ *such that* $\alpha_{i_0}\alpha_n > 0$*;*

(c)
$$\int_{t_1}^{b} \ell(1)(t)\,dt \neq 0.$$

Then the problem $(1), (2)$ *is uniquely solvable.*

Remark 2. Results analogous to Theorems 1–6 can be derived by a standard transformation in the case when $\ell \in \mathcal{N}_{ab}^{-}$, i.e. when ℓ transforms the non-negative non-increasing absolutely continuous functions to the non-positive functions, and when $\ell(1)(t) \le 0$ for a.e. $t \in [a,b]$, respectively.

References

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