The Control Problem of Asynchronous Spectrum of Linear Systems with Depended Blocks of Complete Column Rank

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It is well known that, under certain conditions, a periodic differential system has periodic solutions whose period is incommensurable with the period of the system itself [1–5]. Such solutions are said to be strongly irregular [5]. The conditions of the process in which the oscillations of the system are described by strongly irregular solutions are called an asynchronous mode [6,7], and the frequency spectrum of such solutions is referred to as the asynchronous spectrum. Asynchronous modes of oscillations are implemented in a number of various devices (see, [6] etc.). In particular, there exist systems that transform the energy of a source of high-frequency oscillations into low-frequency ones whose frequency is almost independent of the source frequency. Such systems implement a specifically defined influence on the oscillations, which leads to a periodic transport of energy from an external harmonic source designed for the generation, amplification, or transformation of oscillations. In this case, the oscillatory processes are implemented at the natural frequency of system oscillations, which is not necessarily commensurable with the frequency of the external force. Note that, even in the mid-1930s, the possibility of excitation of oscillations at frequencies with an almost arbitrary relationship with the frequency of changes of parameters was demonstrated in investigation [8] under the supervision by L. I. Mandel'shtam and N. D. Papaleksi of the parametric influence on two-circuit systems.

The problem of synthesis of such modes for linear problems was stated in [9] as a control problem for the asynchronous spectrum. This problem was solved in [10] for linearly independent column basis of some blocks of the coefficient matrix without the average value. In the present paper, we solve the control problem for the asynchronous spectrum with depended blocks of complete column rank.

Consider the linear control system

$$\dot{x} = A(t)x + Bu, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}^n, \quad n \geq 2, \tag{1}$$

where $A(t)$ is a continuous ω-periodic $n \times n$-matrix, B is a constant $n \times n$-matrix. We assume that the control is given in the form of a linear state feedback

$$u = U(t)x \tag{2}$$

with ω-periodic $n \times n$-matrix $U(t)$. The problem of finding the matrix $U(t)$ (the feedback coefficient) such that the closed system

$$\dot{x} = (A(t) + BU(t))x \tag{3}$$

has strongly irregular periodic solutions with a given frequency spectrum L (the objective set) will be called the problem of control of the frequency spectrum of irregular oscillations (asynchronous spectrum)with objective set L.

This problem is a version of the generalization of the spectrum assignment problem in the nonstationary case. Note that, for regular oscillations, the choice of frequencies other than multiples of the frequencies of the right-hand side of system (1) is impossible.

Let $L = \{\lambda_1, \dots, \lambda_r'\}$ be an objective set of frequencies whose elements are pairwise distinct, commensurable with each other, and incommensurable with $2\pi/\omega$. Then there exists a maximum positive real number λ such that $\lambda_1, \dots, \lambda_r'$ are multiples of λ. Set $\Omega = 2\pi/\lambda$ then the ratio ω/Ω is irrational.

Consider the case of a singular matrix B, i.e. $\operatorname{rank} B = r < n \quad (n-r=d)$, and let the first d rows of the matrix B are zero. We denote the matrix consisting of the last r rows of the matrix B by $B_{r,n}$.

We represent the coefficient matrix $A(t)$ in block form corresponding to the structure of the matrix B. Let $A^{(11)}_{d,d}(t)$ and $A^{(21)}_{r,d}(t)$ be its left upper and lower blocks, and let $A^{(12)}_{d,r}(t)$ and $A^{(22)}_{r,r}(t)$ be its right upper and lower blocks. The subscripts show the dimension. In accordance with this representation, in turn, we split the averaged matrix $\widehat{A}$ into four blocks $\widehat{A}^{(11)}_{d,d}$, $\widehat{A}^{(21)}_{r,d}$, $\widehat{A}^{(12)}_{d,r}$, $\widehat{A}^{(22)}_{r,r}$ of the same dimensions. Let $\widetilde{A}(t) = A(t) - \widehat{A}$. Suppose that $\widehat{A}^{(12)}_{d,r} = 0$. Then $\widetilde{A}^{(12)}_{d,r}(t) = A^{(12)}_{d,r}(t)$.

Let us consider the system

$$
\begin{gathered}
\dot{x}^{[d]} = \widehat{A}^{(11)}_{d,d} x^{[d]}, \\
\dot{x}_{[r]} = \big(\widehat{A}^{(21)}_{r,d} + B_{r,n}\widehat{U}_{n,d}\big)x^{[d]} + \big(\widehat{A}^{(22)}_{r,r} + B_{r,n}\widehat{U}_{n,r}\big)x_{[r]}, \\
\widetilde{A}^{(11)}_{d,d}(t)x^{[d]} + A^{(12)}_{d,r}(t)x_{[r]} = 0, \\
(\widetilde{A}^{(21)}_{r,d}(t) + B_{r,n}\widetilde{U}_{n,d}(t))x^{[d]} + (\widetilde{A}^{(22)}_{r,r}(t) + B_{r,n}\widetilde{U}_{n,r}(t))x_{[r]} = 0,
\end{gathered}
\tag{4}
$$

where

$$
\begin{gathered}
x = \operatorname{col}(x^{[d]}, x_{[r]}), \quad x^{[d]} = \operatorname{col}(x_1, \dots, x_d), \quad x_{[r]} = \operatorname{col}(x_{d+1}, \dots, x_n), \\
\widehat{U} = \big\{\widehat{U}_{n,d}, \widehat{U}_{n,r}\big\}, \quad \widetilde{U}(t) = \big\{\widetilde{U}_{n,d}(t), \widetilde{U}_{n,r}(t)\big\}.
\end{gathered}
$$

Suppose that the upper left and right blocks of the matrix $\widetilde{A}(t)$ have complete column rank

$$
\operatorname{rank}_{\rm col} \widetilde{A}^{(11)}_{d,\,d} = d, \quad \operatorname{rank}_{\rm col} A^{(12)}_{d,\,r} = r \tag{5}
$$

and are linearly depended

$$
\widetilde{A}^{(11)}_1(t)F_1 + \widetilde{A}^{(11)}_2(t)F_2 = A^{(12)}_{d,r}(t)G, \tag{6}
$$

where $\widetilde{A}^{(11)}_{d,d}(t) = \{\widetilde{A}^{(11)}_1(t), \widetilde{A}^{(11)}_2(t)\}$, F_1 is nonsingular $s \times s$-martix, F_2 is $(d-s) \times s$-martix. G is a constant $r \times s$ matrix $(1 \le s \le \min\{d, r\})$.

Let $x'^{[d]} = \operatorname{col}(x_1, y_2, \dots, y_s)$, $x''^{[d]} = \operatorname{col}(x_{s+1}, \dots, x_d)$. In accordance with the representation of the vector $x[d]$ via $x'^{[d]}$ and $x''^{[d]}$ on the basis of the matrix $\widehat{A}^{(11)}_{d,d}$ we form four matrices $\widehat{A}^{(11)}_1$, $\widehat{A}^{(11)}_2$, $\widehat{A}^{(21)}_1$, and $\widehat{A}^{(22)}_1$.

By assumption (5), (6), system (4) can be represented in the form

$$
\begin{gathered}
\dot{x}'^{[d]} = (\widehat{A}^{(11)}_1 + \widehat{A}^{(11)}_2 H)x'^{[d]}, \\
H\dot{x}'^{[d]} = (\widehat{A}^{(11)}_3 + \widehat{A}^{(11)}_4 H)x'^{[d]}, \\
\dot{x}_{[r]} = (\widehat{A}^{(21)}_{r,d} + B_{r,n}\widehat{U}_{n,d})x^{[d]} + (\widehat{A}^{(22)}_{r,r} + B_{r,n}\widehat{U}_{n,r})x_{[r]}, \\
(\widetilde{A}^{(21)}_{r,d}(t) + B_{r,n}\widetilde{U}_{n,d}(t))x^{[d]} + (\widetilde{A}^{(22)}_{r,r}(t) + B_{r,n}\widetilde{U}_{n,r}(t))x_{[r]} = 0, \\
x''^{[d]} = Hx'^{[d]}, \quad x_{[r]} = Px'^{[d]} \quad (H = F_2F_1^{-1},\ P = -GF_1^{-1}),
\end{gathered}
\tag{7}
$$

where

$$
x'^{[d]} = \operatorname{col}(x_1, y_2, \dots, y_s), \quad x''^{[d]} = \operatorname{col}(x_{s+1}, \dots, x_d).
$$

It follows from [4] that systems (3) and (7) are equivalent in the sense of existence of strongly irregular periodic solutions. Therefore, a strongly irregular solution of the closed system (3) is a trigonometric polynomial.

The following assertion holds.

Theorem. *Let the first d rows in the matrix B be zero, and the remaining r rows be linearly independed. Suppose that $\widehat{A}^{(12)}_{d,r}=0$. Under assumptions* (5), (6), *the control problem for an asynchronous spectrum for system* (1) *can be reduced to finding constant matrices $\widehat{U}_{n,d}$, $\widehat{U}_{n,r}$ and ω-periodic matrices $\widetilde{U}_{n,d}(t)$, $\widetilde{U}_{n,r}(t)$ such that system* (7) *has an Ω-periodic solution* $\operatorname{col}(x'^{[d]}(t),x''^{[d]}(t),x_{[r]}(t))$ *whose frequencies form the objective set L. A strongly irregular solution of the closed system* (3) *is a trigonometric polynomial.*

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