Multipoint Boundary Value Problem for the Linear Matrix Lyapunov Equation with Parameter

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This work is a continuation and development of [1] and the problem is investigated with the help of constructive regularization method [2, Ch. 1].

Consider the multipoint boundary value problem for the matrix equation

$$\frac{dX}{dt} = \big(A_0(t) + \lambda A_1(t)\big)X + XB(t) + F(t), \quad X \in \mathbb{R}^{n\times m} \tag{1}$$

with the condition

$$\sum_{i=1}^{k} M_i X(t_i) = 0, \quad 0 = t_1 < t_2 < \cdots < t_k = \omega, \tag{2}$$

where $A_0(t)$, $A_1(t)$, $B(t)$, $F(t)$ are matrices for class $\mathbb{C}[0,\omega]$ of corresponding dimensions, M_i are given constant $(n \times n)$-matrices, $\lambda \in \mathbb{R}$.

A nonlinear problem of the type (1), (2) was studied by qualitative methods in [3].

We investigate the problem (1), (2) on the bases of the method of integral equations. We use the additive decomposition of the matrix $B(t)$ in the form $B(t) = B_1(t) + B_2(t)$, where the matrices $B_1(t)$, $B_2(t)$ are chosen in a certain way (see, for example, [2, Ch. 1]).

We introduce the following notations.

$$\gamma = \|\Phi^{-1}\|, \ \ \mu_1 = \max_t \|V(t)\|, \ \ \mu_2 = \max_t \|V^{-1}(t)\|, \ \ v_i = \|V_i\|, \ \ m_i = \|M_i\|, \ \ \varepsilon = |\lambda|,$$

$$\beta_2 = \max_t \|B_2(t)\|, \ \ \alpha_i = \max_t \|A_i(t)\| \ \ (i = 0, 1), \quad q_0 = \gamma\mu_1\mu_2(\alpha_0 + \beta_2)\omega \sum_{i=1}^{k} m_i v_i,$$

$$q_1 = \gamma\mu_1\mu_2\alpha_1\omega \sum_{i=1}^{k} m_i v_i, \quad N = \gamma\mu_1\mu_2\omega h \sum_{i=1}^{k} m_i v_i,$$

where Φ is a linear operator: $\Phi Y \equiv \sum\limits_{i=1}^{k} M_i Y V_i$; $V_i = V(t_i)$, $V(t)$ is a fundamental matrix of the equation $dV/dt = VB_1(t)$; $\|\bullet\|$ is an agreement matrix norm.

Theorem. *Let the operator Φ be invertible and $q_0 < 1$. Then for $|\lambda| < (1 - q_0)/q_1$ the problem (1), (2) is uniquely solvable; its solution $X(t)$ can be represented as the limit of a uniformly convergent sequence of matrix functions defined by an integral recursion relation and satisfying the condition (2); moreover, the following estimate holds*

$$\|X(t,\lambda)\| \le \frac{N}{1 - q_0 - \varepsilon q_1}\,. \tag{3}$$

Proof. We use a constructive method that follows from the approach in [2]. Then we have equivalent integral equation

$$X(t) = \bigg(\Phi^{-1}\Big\{\sum_{i=1}^{k} M_i \int\limits_{t_i}^{t} \Big[A(\tau)X(\tau) + X(\tau)B_2(\tau) + F(\tau)\Big]V^{-1}(\tau)\,d\tau \cdot V_i\Big\}\bigg)V(t), \tag{4}$$

where $X(t) \equiv X(t,\lambda)$, $A(\tau) \equiv A_0(\tau) + \lambda A_1(\tau)$.

To analyze the solvability of the matrix equation (4), we use the contraction mapping principle [4, p. 605]. Next, we obtain an integral recursion relation for the approximate solution

$$X_p(t) = \bigg(\Phi^{-1}\bigg\{\sum_{i=1}^{k} M_i \int_{t_i}^{t} \Big[A(\tau)X_{p-1}(\tau) + X_{p-1}(\tau)B_2(\tau) + F(\tau)\Big]V^{-1}(\tau)\,d\tau \cdot V_i\bigg\}\bigg)V(t), \quad p = 1,2,\dots\,. \tag{5}$$

For the initial approximation $X_0(t)$ one can take any matrix of the class $\mathbb{C}(I,\mathbb{R}^{n\times n})$.

We proof next: the functions $X_1(t), X_2(t), \dots$ satisfy the condition (2). Consider the algorithm (5) in differential form:

$$\begin{aligned}
\frac{dX_p(t)}{dt} &= X_p(t)B_1(t) + \bigg(\Phi^{-1}\bigg\{\sum_{i=1}^{k} M_i\Big[A(t)X_{p-1}(t) + X_{p-1}(t)B_2(t) + F(t)\Big]V^{-1}(t)V_i\bigg\}\bigg)V(t) = \\
&= X_p(t)B_1(t) + \Big(\Phi^{-1}\Big\{\Phi\big[A(t)X_{p-1}(t) + X_{p-1}(t)B_2(t) + F(t)\big]V^{-1}(t)\Big\}\Big)V(t) = \\
&= X_p(t)B_1(t) + \big[A(t)X_{p-1}(t) + X_{p-1}(t)B_2(t) + F(t)\big]V^{-1}(t)V(t) = \\
&= X_p(t)B_1(t) + \big[A(t)X_{p-1}(t) + X_{p-1}(t)B_2(t) + F(t)\big].
\end{aligned}$$

Hence we obtain the representation

$$\frac{dX_p(t)}{dt} = X_p(t)B_1(t) + \big[A(t)X_{p-1}(t) + X_{p-1}(t)B_2(t) + F(t)\big]. \tag{6}$$

From (6) we have

$$\big[A(\tau)X_{p-1}(\tau) + X_{p-1}(\tau)B_2(\tau) + F(\tau)\big]\,d\tau = dX_p(\tau) - X_p(\tau)B_1(\tau)\,d\tau. \tag{7}$$

By using (7), on the bases of (6) we obtain

$$\begin{aligned}
X_p(t) &= \bigg(\Phi^{-1}\bigg\{\sum_{i=1}^{k} M_i \int_{t_i}^{t} \big[dX_p(\tau) - X_p(\tau)B_1(\tau)\,d\tau\big]V^{-1}(\tau)\cdot V_i\bigg\}\bigg)V(t) = \\
&= \bigg(\Phi^{-1}\bigg\{\sum_{i=1}^{k} M_i \int_{t_i}^{t} (dX_p(\tau))V^{-1}(\tau)V_i - \sum_{i=1}^{k} M_i \int_{t_i}^{t} X_p(\tau)B_1(\tau)V^{-1}(\tau)\,d\tau \cdot V_i\bigg\}\bigg)V(t) = \\
&= \bigg(\Phi^{-1}\bigg\{\sum_{i=1}^{k} M_i\bigg(X_p(\tau)V^{-1}(\tau)\Big|_{t_i}^{t} + \int_{t_i}^{t} X_p(\tau)B_1(\tau)V^{-1}(\tau)\,d\tau\bigg)V_i - \\
&\qquad - \sum_{i=1}^{k} M_i \int_{t_i}^{t} X_p(\tau)B_1(\tau)V^{-1}(\tau)\,d\tau \cdot V_i\bigg\}\bigg)V(t) = \\
&= \bigg(\Phi^{-1}\bigg\{\sum_{i=1}^{k} M_i\bigg(X_p(t)V^{-1}(t) - X_p(t_i)V^{-1}(t_i) + \int_{t_i}^{t} X_p(\tau)B_1(\tau)V^{-1}(\tau)\,d\tau\bigg)V_i - \\
&\qquad - \sum_{i=1}^{k} M_i \int_{t_i}^{t} X_p(\tau)B_1(\tau)V^{-1}(\tau)\,d\tau \cdot V_i\bigg\}\bigg)V(t) =
\end{aligned}$$

$$= \left(\Phi^{-1}\left\{\sum_{i=1}^{k}\left(M_iX_p(t)V^{-1}(t)V_i - M_iX_p(t_i) + M_i\int_{t_i}^{t} X_p(\tau)B_1(\tau)V^{-1}(\tau)\,d\tau\cdot V_i\right) -\right.\right.$$
$$\left.\left. - \sum_{i=1}^{k} M_i\int_{t_i}^{t} X_p(\tau)B_1(\tau)V^{-1}(\tau)\,d\tau\cdot V_i\right\}\right)V(t) =$$
$$= \left(\Phi^{-1}\left\{\sum_{i=1}^{k} M_iX_p(t)V^{-1}(t)V_i - \sum_{i=1}^{k} M_iX_p(t_i) + \sum_{i=1}^{k} M_i\int_{t_i}^{t} X_p(\tau)B_1(\tau)V^{-1}(\tau)\,d\tau\cdot V_i -\right.\right.$$
$$\left.\left. - \sum_{i=1}^{k} M_i\int_{t_i}^{t} X_p(\tau)B_1(\tau)V^{-1}(\tau)\,d\tau\cdot V_i\right\}\right)V(t) =$$
$$= \left(\Phi^{-1}\left\{\Phi\big[X_p(t)V^{-1}(t)\big] - \sum_{i=1}^{k} M_iX_p(t_i)\right\}\right)V(t) =$$
$$= \left(\Phi^{-1}\Phi\big[X_p(t)V^{-1}(t)\big] - \Phi^{-1}\sum_{i=1}^{k} M_iX_p(t_i)\right)V(t) =$$
$$= \left(X_p(t)V^{-1}(t) - \Phi^{-1}\sum_{i=1}^{k} M_iX_p(t_i)\right)V(t) = X_p(t) - \left(\Phi^{-1}\sum_{i=1}^{k} M_iX_p(t_i)\right)V(t). \tag{8}$$

Note that the formula (8) yields

$$\sum_{i=1}^{k} M_iX_p(t_i) = 0.$$

Let us analyze the convergence of the sequence $\{X_p(t)\}_1^{\infty}$. By (5), we have

$$X_{p+1}(t) - X_p(t) = \mathfrak{L}(X_p) - \mathfrak{L}(X_{p-1}),\ \ p = 1, 2, \dots, \tag{9}$$

where

$$\mathfrak{L}(Y) = \left(\Phi^{-1}\left\{\sum_{i=1}^{k} M_i\int_{t_i}^{t}\Big[A(\tau)Y(\tau) + Y(\tau)B_2(\tau) + F(\tau)\Big]V^{-1}(\tau)\,d\tau\cdot V_i\right\}\right)V(t).$$

By estimating the norm in (9), we obtain the inequality

$$\|X_p - X_{p-1}\|_C \le q^p\|X_1 - X_0\|_C,\ \ p = 1, 2, \dots, \tag{10}$$

where $q = q_0 + \varepsilon q_1$, $\|X_1 - X_0\|_C = \|\mathfrak{L}(X_0) - X_0\|_C$.

By using (10), one can show that the sequence converges uniformly with respect to $t \in [0, \omega]$ to a solution of the integral equation (4), equivalent to the problem (1), (2), and we obtain the estimates

$$\|X - X_r\|_C \le \frac{q^r}{1-q}\,\|X_1 - X_0\|_C,\ \ r = 0, 1, 2, \dots,$$
$$\|X\|_C \le \|X_0\|_C + \frac{\|X_1 - X_0\|_C}{1-q}\,. \tag{11}$$

From (5) we have the estimate $\|X_1\|_C \le N$ for $X_0 = 0$, and from (11) we have the inequality (3). $\square$

References

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