Asymptotic Behavior of Solutions with Slowly Varying Derivatives of Essentially Nonlinear Second Order Differential Equations

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The differential equation

$$y'' = \alpha_0 p(t)\varphi_0(y)\exp(R(|\ln|y||))\varphi_1(y'), \tag{1}$$

where $\alpha_0 \in \{-1, 1\}$, $p : [a, \omega[\to]0, +\infty[$ [1] $(-\infty < a < \omega \le +\infty)$, $\varphi_i : \Delta_{Y_i} \to]0, +\infty[$ $(i = 0, 1)$, are continuous functions, $R :]0, +\infty[\to]0, +\infty[$ is a continuously differentiable function, $Y_i \in \{0, \pm\infty\}$, Δ_{Y_i} is either the interval $[y_i^0, Y_i[$ [2], or the interval $]Y_i, y_i^0]$, is considered.

We suppose also that R is a regularly varying function of index μ, every $\varphi_i(z)$ is regularly varying as $z \to Y_i$ $(z \in \Delta_{Y_i})$ of index σ_i and $0 < \mu < 1$, $\sigma_0 + \sigma_1 \ne 1$.

We call the measurable function $\varphi : \Delta_Y \to]0, +\infty[$ a regularly varying as $z \to Y$ of index σ if for every $\lambda > 0$ we have

$$\lim_{\substack{z \to Y \\ z \in \Delta_Y}} \frac{\varphi(\lambda z)}{\varphi(z)} = \lambda^\sigma.$$

Here $Y \in \{0, \pm\infty\}$, Δ_Y is some one-sided neighbourhood of Y. If $\sigma = 0$, such function is called slowly varying.

It follows from the results of the monograph [1] that regularly varying functions have the next properties.

M_1: The function $\varphi(z)$ is regularly varying of index σ as $z \to Y$ if and only if it admits the representation

$$\varphi(z) = z^\sigma \theta(z),$$

where $\theta(z)$ is a slowly varying function as $z \to Y$.

M_2: If the function $L : \Delta_{Y^0} \to]0, +\infty[$ is slowly varying as $z \to Y_0$, the function $\varphi : \Delta_Y \to \Delta_{Y^0}$ is regularly varying as $z \to Y$, then the function $L(\varphi) : \Delta_Y \to]0, +\infty[$ is slowly varying as $z \to Y$.

M_3: If the function $\varphi : \Delta_Y \to]0, +\infty[$ satisfies the condition

$$\lim_{\substack{z \to Y \\ z \in \Delta}} \frac{z\varphi'(z)}{\varphi(z)} = \sigma \in \mathbb{R},$$

then φ is regularly varying as $z \to Y$of index σ.

We call the solution y of the equation (1) the $P_\omega(Y_0, Y_1, \lambda_0)$-solution, where $-\infty \le \lambda_0 \le +\infty$, if the following conditions take place

$$y^{(i)} : [t, \omega[\to \Delta_{Y_i}, \quad \lim_{t \uparrow \omega} y^{(i)}(t) = Y_i \ \ (i = 0, 1), \quad \lim_{t \uparrow \omega} \frac{(y'(t))^2}{y''(t)\, y(t)} = \lambda_0. \tag{2}$$

[1] If $\omega > 0$, we take $a > 0$.

[2] If $Y_i = +\infty$ $(Y_i = -\infty)$, we take $y_i^0 > 0$ $(y_i^0 < 0)$.

All $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of the equation (1) were investigated in [2, 3] for $\lambda_0\in\mathbb{R}\setminus\{0\}$. The necessary and sufficient conditions for the existence and asymptotic representations of such solutions as $t\uparrow\omega$ were found. The cases $\lambda_0\in\{0,\pm\infty\}$ are singular in studying of $P_\omega(Y_0,Y_1,\lambda_0)$-solutions of (1). To investigate such solutions we must put additional conditions to the right side of equation (1).

We say that a slowly varying as $z\to Y$ $(z\in\Delta_Y)$ function $\theta:\Delta_Y\to]0;+\infty[$ satisfies the condition S if for any continuous differentiable function $L:\Delta_{Y_i}\to]0;+\infty[$ such that

$$\lim_{\substack{z\to Y_i\\ z\in\Delta_{Y_i}}}\frac{zL'(z)}{L(z)}=0,$$

the next condition takes place

$$\theta(zL(z))=\theta(z)(1+o(1))\ \text{ as }\ z\to Y\ \ (z\in\Delta_Y).$$

By the statement M_1 and definition of φ_0 it is clear that $\varphi_0(z)|z|^{-\sigma_0}$ is slowly varying function as $z\to Y_0$ $(z\in\Delta_{Y_0})$. The sufficiently important class of $P_\omega(Y_0,Y_1,\infty)$-solutions of the equation (1) was investigated only in cases, when $R(z)\equiv 1$ and the function $\varphi_0(z)|z|^{-\sigma_0}$ satisfies the condition S. Using (2) and statements M_1–M_3, it is easy to see that the first derivative of every $P_\omega(Y_0,Y_1,\infty)$-solution of the equation (1) is a slowly varying function as $t\uparrow\omega$. This is one of the most difficult problems in studying such solutions. For equations of the type (1) that contain, for example, functions like $\exp(\sqrt{|\ln|y||})$ or $\exp(\sqrt[m]{|\ln\|y\||})$, the asymptotic representations of $P_\omega(Y_0,Y_1,\infty)$-solutions were not established before. The aim of the work is to establish the necessary and sufficient conditions for the existence and asymptotic representations as $t\uparrow\omega$ of $P_\omega(\lambda^0_{n-1})$-solutions of the equation (1) in general case. Let us note that the function $\exp(R(|\ln|z||))$ does not satisfy the condition S.

We need the following subsidiary notations

$$\pi_\omega(t)=\begin{cases} t & \text{as } \omega=+\infty,\\ t-\omega & \text{as } \omega<+\infty,\end{cases}\qquad \theta_0(z)=\Psi_0(z)|z|^{-\sigma_0}.$$

We put also

$$L(t)=p(t)|\pi_\omega(t)|^{\sigma_0+1}\theta_0\big(|\pi_\omega(t)|\operatorname{sign}y_0^0\big),$$

$$I_0(t)=\int_{A^0_\omega}^{t} p(\tau)|\pi_\omega(\tau)|^{\sigma_0}\theta_0\big(|\pi_\omega(\tau)|\operatorname{sign}y_0^0\big)\,d\tau,$$

$$A^0_\omega=\begin{cases} a, & \text{if } \displaystyle\int_a^\omega p(t)|\pi_\omega(t)|^{\sigma_0}\theta_0\big(|\pi_\omega(t)|\operatorname{sign}y_0^0\big)\,dt=+\infty,\\ \omega, & \text{if } \displaystyle\int_a^\omega p(t)|\pi_\omega(t)|^{\sigma_0}\theta_0\big(|\pi_\omega(t)|\operatorname{sign}y_0^0\big)\,dt<+\infty,\end{cases}$$

in case $\lim\limits_{t\uparrow\omega}|\pi_\omega(\tau)|\operatorname{sign}y_0^0=Y_0$. Here we choose $b\in[a,\omega[$ so that $|\pi_\omega(t)|\operatorname{sign}y_0^0\in\Delta_{Y_0}$ as $t\in[b,\omega[$.

The following conclusions are valid for the equation (1).

Theorem 1. *The following conditions are necessary for the existence of the $P_\omega(Y_0,Y_1,\pm\infty)$-solutions of the equation* (1)

$$Y_0=\begin{cases}\pm\infty, & \text{if } \omega=+\infty,\\ 0, & \text{if } \omega<+\infty,\end{cases}\qquad \pi_\omega(t)y_0^0y_1^0>0\ \text{ as }\ t\in[a,\omega[. \tag{3}$$

If the function $\varphi_0(z)|z|^{\sigma_0}$ *satisfies the condition* S *and the statement*

$$\lim_{t\uparrow\omega}\frac{R'(|\ln|\pi_\omega(t)||)I_0(t)}{\pi_\omega(t)I_0'(t)}=0 \tag{4}$$

is true, then the conditions (3) *and*

$$\alpha_0 y_1^0(1-\sigma_0-\sigma_1)I_0(t)>0 \ \text{ as } \ t\in[b,\omega[\,,$$
$$\lim_{t\uparrow\omega} y_1^0|I_0(t)|^{\frac{1}{1-\sigma_0-\sigma_1}}=Y_1, \quad \lim_{t\uparrow\omega}\frac{\pi_\omega(t)I_0'(t)}{I_0(t)}=0$$

are necessary and sufficient for the existence of $P_\omega(Y_0,Y_1,\pm\infty)$*-solutions of the equation* (1)*. For any such solution the following asymptotic representations take place as* $t\uparrow\omega$*:*

$$\frac{y'(t)|y'(t)|^{-\sigma_0}}{\varphi_1(y'(t))\exp(R(|\ln|y(t)||))}=\alpha_0(1-\sigma_0-\sigma_1)I_0(t)[1+o(1)],$$
$$\frac{y'(t)}{y(t)}=\frac{1}{\pi_\omega(t)}\,[1+o(1)].$$

Theorem 2. *Let the function* $\varphi_0(z)|z|^{\sigma_0}$ *satisfy the condition* S*, but the statement* (4) *do not fulfilled. If*

$$\lim_{t\uparrow\omega}\frac{R'(|\ln|\pi_\omega(t)||)L(t)}{\pi_\omega(t)L'(t)}=\infty,$$

then the conditions (3) *and*

$$\alpha_0 y_1^0(1-\sigma_0-\sigma_1)\ln|\pi_\omega(t)|>0 \ \text{ for } \ t\in[a,\omega[\,,$$
$$\lim_{t\uparrow\omega} y_1^0\exp\Big(\frac{1}{1-\sigma_0-\sigma_1}R(|\ln|\pi_\omega(t)||)\Big)=Y_1$$

are necessary and sufficient for the existence of $P_\omega(Y_0,Y_1,\pm\infty)$*-solutions of the equation* (1)*. For any such solution the following asymptotic representations take place as* $t\uparrow\omega$*:*

$$\frac{|y'(t)|^{1-\sigma_0}}{\varphi_1(y'(t))\exp(R(|\ln|y(t)||))}=\frac{|1-\sigma_0-\sigma_1|L(t)}{R'(|\ln|\pi_\omega(t)||)}\,[1+o(1)],$$
$$\frac{y'(t)}{y(t)}=\frac{1}{\pi_\omega(t)}\,[1+o(1)].$$

References

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