Kiguradze-type and Belohorec-type Oscillation Theorems for Second Order Nonlinear Dynamic Equations on Time Scales

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Consider the second order nonlinear dynamic equations

$$x^{\Delta\Delta}+p(t)x^{\alpha}(\sigma(t))=0, \tag{1}$$

where $p\in C(\mathbb{T},R)$, $t\in\mathbb{T}$ is a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) with $\sup\mathbb{T}=\infty$, $\sigma(t)=\inf\{s\in\mathbb{T}:\ s>t\}$ and $\alpha\neq 1$, $\alpha>0$ is the quotient of odd positive integers. Equation (1) is called superlinear if $\alpha>1$ and sublinear if $0<\alpha<1$. We call an equation oscillatory if all its continuable solutios are oscillatory.

When $\mathbb{T}=\mathbb{R}$, the dynamic equation (1) is the second order nonlinear differential equation

$$x''(t)+p(t)x^{\alpha}(t)=0. \tag{2}$$

When $\mathbb{T}=\mathbb{N}_0$, the dynamic equation (1) is the second order nonlinear difference equation

$$\Delta^2x(n)+p(t)x^{\alpha}(n+1)=0. \tag{3}$$

When $p(t)$ is nonnegative, stronger oscillation results exist for the nonlinear equation (2) when $\alpha\neq 1$, notably the following:

Theorem 1 (Atkinson [2])**.** *Let $\alpha>1$. Then* (2) *is oscillatory if and only if*

$$\int^{\infty} tp(t)\,dt=\infty. \tag{4}$$

Theorem 2 (Belohorec [10])**.** *Let $0<\alpha<1$. Then* (2) *is oscillatory if and only if*

$$\int^{\infty} t^{\alpha}p(t)\,dt=\infty. \tag{5}$$

When $p(t)$ is allowed to take on negative values, for $\alpha>1$, Kiguradze [1] proved that (4) is sufficient for the differential equation (2) to be oscillatory and for $0<\alpha<1$ Belohorec [11] proved that (5) is a sufficient for the differential equation (2) to be oscillatory. These results have been further extended by Kwong and Wong [12].

When $p(n)$ is nonnegative, J. W. Hooker and W. T. Patula [5, Theorem 4.1], A. Mingarelli [6], respectively proved that

Theorem 3. *Let $\alpha>1$. Then* (3) *is oscillatory if and only if*

$$\sum_{1}^{\infty}(n+1)p(n)=\infty. \tag{6}$$

Theorem 4. *Let $0 < \alpha < 1$. Then* (3) *is oscillatory if and only if*

$$\sum_{1}^{\infty}(n+1)^{\alpha}p(n)=\infty.$$

In this paper, when $p(t)$ is allowed to take on negative values, we obtain the following results.

Theorem A. *Let $\alpha > 1$ and there exist a real number β, $0 < \beta \le 1$ such that $\int_{t_0}^{\infty}(\sigma(t))^{\beta}p(t)\,\Delta t=\infty$. Then* (1) *is oscillatory.*

Theorem B. *Let $0<\alpha<1$ and there exist a real number β, $0<\beta\le 1$ such that $\int_{t_0}^{\infty}(\sigma(t))^{\alpha\beta}p(t)\,\Delta t=\infty$. Then* (1) *is oscillatory.*

From Theorem A and Theorem B, we can get the following corollaries.

Corollary 5. *Let $\alpha > 1$ and $p(t)$ be allowed to take on negative values. Then* (3) *is oscillatory if*

$$\sum_{1}^{\infty}(n+1)p(n)=\infty.$$

Corollary 6. *Let $0 < \alpha < 1$ and $p(t)$ be allowed to take on negative values. Then* (3) *is oscillatory if*

$$\sum_{1}^{\infty}(n+1)^{\alpha}p(n)=\infty.$$

Example 7. The superlinear difference equation

$$\Delta^2x(n)+\Big[\frac{a}{(n+1)^{b+1}}+\frac{c(-1)^n}{(n+1)^b}\Big]x^{\alpha}(n+1)=0,\ \ \alpha>1,$$

for $a > 0$, $0 < b \le 1$, is oscillatory. In [3], this result is shown to be true only for $0 < b < 1$ and $0 < bc < a < c(1-b)$.

Example 8. The sublinear difference equation

$$\Delta^2x(n)+\Big[\frac{1}{(n+1)^{c+1}}+\frac{b(-1)^n}{(n+1)^c}\Big]x^{\alpha}(n+1)=0,\ \ 0<\alpha<1,$$

is oscillatory if $0 \le c \le \alpha$, and is nonoscillatory if $c > \alpha$ (using Theorem 2.1 in [7]).

To prove Theorem A and Theorem B, we need the following Lemmas.

Lemma 9. *Suppose that $\alpha > 1$ and $x(t) > 0$ for $t \in [T,\infty)_{\mathbb{T}}$. Then we have*

$$\int_{T}^{t}\frac{x^{\Delta}(s)}{x^{\alpha}(\sigma(s))}\,\Delta s\le\frac{x^{-\alpha+1}(T)}{\alpha-1}.$$

Proof. Using the Pötzsche chain rule [4, Theorem 1.90], we get that

$$\begin{aligned}\Big(\frac{x^{-\alpha+1}(s)}{\alpha-1}\Big)^{\Delta}&=-\int_{0}^{1}\frac{dh}{(x(s)+h\mu(s)x^{\Delta}(s))^{\alpha}}\,x^{\Delta}(s)=\\&=-\int_{0}^{1}\frac{dh}{(hx(\sigma(s))+(1-h)x(s))^{\alpha}}\,x^{\Delta}(s).\end{aligned}\tag{7}$$

When $x^{\Delta}(s) \geq 0$, that is $x(\sigma(s)) \geq x(s)$, from (7) we have

$$\Big(\frac{x^{-\alpha+1}(s)}{\alpha-1}\Big)^{\Delta} \leq -\int\limits_0^1 \frac{dh}{(hx(\sigma(s))+(1-h)x(\sigma(s)))^{\alpha}}\, x^{\Delta}(s) = -\frac{x^{\Delta}(s)}{x^{\alpha}(\sigma(s))}\,. \tag{8}$$

When $x^{\Delta}(s) \leq 0$, that is $x(\sigma(s)) \leq x(s)$, from (7) we also have

$$\Big(\frac{x^{-\alpha+1}(s)}{\alpha-1}\Big)^{\Delta} \leq -\int\limits_0^1 \frac{dh}{(hx(\sigma(s))+(1-h)x(\sigma(s)))^{\alpha}}\, x^{\Delta}(s) = -\frac{x^{\Delta}(s)}{x^{\alpha}(\sigma(s))}\,. \tag{9}$$

So from (8) and (9), we get that for $s \in [T,\infty)_{\mathbb{T}}$

$$\Big(\frac{x^{-\alpha+1}(s)}{\alpha-1}\Big)^{\Delta} \leq -\frac{x^{\Delta}(s)}{x^{\alpha}(\sigma(s))}\,. \tag{10}$$

Integrating (10) from T to t, we get

$$\int\limits_T^t \frac{x^{\Delta}(s)}{x^{\alpha}(\sigma(s))}\,\Delta s \leq -\int\limits_T^t \Big(\frac{x^{-\alpha+1}(s)}{\alpha-1}\Big)^{\Delta}\Delta s = \frac{x^{-\alpha+1}(T)}{\alpha-1} - \frac{x^{-\alpha+1}(t)}{\alpha-1} \leq \frac{x^{-\alpha+1}(T)}{\alpha-1}\,. \qquad \square$$

Similarly, we have

Lemma 10. *Suppose that* $0 < \alpha < 1$ *and* $x(t) > 0$ *for* $t \in [T,\infty)_{\mathbb{T}}$. *Then we have*

$$\int\limits_T^t \frac{x^{\Delta}(s)}{x^{\alpha}(s)}\,\Delta s \geq -\frac{x^{1-\alpha}(T)}{1-\alpha}\,,$$

and

$$\int\limits_T^t \frac{(x^{\alpha}(s))^{\Delta}x(\sigma(s))}{x^{\alpha}(s)x^{\alpha}(\sigma(s))}\,\Delta s \geq -\frac{\alpha x^{1-\alpha}(T)}{1-\alpha}\,.$$

The complete proofs of Theorem A and Theorem B are in [8] and [9].

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