On Asymptotic Behavior of Solutions to Nonlinear Differential Equations with a Small Right-Hand Side

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1 Introduction

The problem of asymptotic behavior of solutions to nonlinear differential equations with an exponentially small or power-law small right-hand sides is investigated.

Consider the equation

$$y^{(n)} + p(x)|y|^k \operatorname{sgn} y = F(x), \quad n \geq 2, \quad k > 1, \tag{1}$$

with continuous functions $p(x)$ and $F(x)$.

Equation (1) with $F(x) = 0$ was investigated from different points of view (see, for example, [8], [4] and the bibliography therein). In particular, the asymptotic behavior of its solutions vanishing at infinity is described. If the function $F(x)$ is sufficiently small, it is possible to describe the asymptotic behavior of vanishing at infinity solutions to equation (1), too. Previous results are published in [1]– [6]. Results of this type for ordinary differential equations and their systems can be useful also to investigate some problems for partial differential equations (see, for example, [7]).

Note that there exist notions of asymptotic equivalence different from the one used here (cf. [10]–[17]).

2 Main results

In this section results on asymptotic equivalence of solutions to differential equations with different right-hand sides are formulated.

1 Exponentially equivalent right-hand sides

Theorem 2.1 (see [6]). *Let $f(x)$, $g(x)$, and $p(x)$ be bounded continuous functions defined in a neighborhood of $+\infty$. Suppose $y(x)$ is a solution to the equation*

$$y^{(n)} + p(x)|y|^k \operatorname{sgn} y = f(x)\, e^{-\beta x} \tag{2}$$

with $n \geqslant 2$, $k > 1$, $\beta > 0$ and $y(x) \to 0$ as $x \to +\infty$. Then there exists a unique solution $z(x)$ to the equation

$$z^{(n)} + p(x)|z|^k \operatorname{sgn} z = g(x)\, e^{-\beta x} \tag{3}$$

such that $|z(x) - y(x)| = O(e^{-\beta x})$ as $x \to +\infty$.

To prove this result we use the following lemmas.

Lemma 2.1. *If a function $y(x)$ and its n-th derivative $y^{(n)}(x)$ both tend to zero as $x \to +\infty$, then the same is true for all of its lower-order derivatives $y^{(j)}(x)$, $0 < j < n$.*

Lemma 2.2. *Suppose a function $y(x)$ satisfies the inequality $|y^{(j)}(x)| \geq W > 0$ on a segment I of length Δ. Then there exists a segment $I' \subset I$ of length $4^{-j}\Delta$ with $|y(x)| \geq W(2^{-1-j}\Delta)^j$ satisfied for all $x \in I'$.*

Lemma 2.3. *Let $y(x)$ be a solution to equation* (2) *tending to zero as $x \to +\infty$. Then*

$$y(x) = \mathbf{J}^n \big[e^{-\beta x} f(x) - p(x) |y(x)|^k \operatorname{sgn} y(x) \big],$$

where the operator $\mathbf{J}$ takes each sufficiently rapidly decreasing function $\varphi(x)$ to its primitive function vanishing at infinity:

$$\mathbf{J}[\varphi](x) = -\int\limits_x^\infty \varphi(\xi)\, d\xi.$$

Corollary 2.1. *Suppose the function $F(x)$ in equation* (1) *satisfies the condition*

$$|F(x)| \le Ce^{-\beta x}, \ \ C > 0, \ \ \beta > 0, \tag{4}$$

and $p(x)$ is a bounded continuous function. Then for any solution $y(x)$ to equation (1) *tending to zero as $x \to \infty$ there exists a solution $z(x)$ to equation* (1) *with F(x)=0 such that*

$$|y(x) - z(x)| = O(e^{-\beta x}), \ \ x \to \infty.$$

Remark 2.1. Note that if $p(x) \to p_0 \neq 0$ as $x \to \infty$, for $n = 2$ [8] and $n \in \{3, 4\}$ ([3] and [4], Ch.I, Section 5.4) asymptotic behavior of all solutions to equation (1) with $F(x) = 0$ is described. In particular, if $(-1)^n p_0 < 0$, then all nontrivial vanishing at infinity solutions $z(x)$ to equation (1) with $F(x) = 0$ satisfy

$$z(x) = C\, x^{-\alpha} (1 + o(1)), \ \ x \to \infty, \ \ \text{with} \alpha = \frac{n}{k-1}, \quad C = \Big(\frac{1}{p_0} \prod_{j=0}^{n-1} (\alpha + j) \Big)^{\frac{1}{k-1}}.$$

As for $n \ge 5$, solutions with the above asymptotic behavior also exist if $p(x)$ tends to p_0 quickly enough. This was proved in [4] (Ch.I, Theorem 5.3) for the function p depending on $x, y, y', \dots, y^{(n-1)}$ and satisfying rather cumbersome conditions, which are reduced, in the case $p(x)$, to the condition $p(x) = p_0 + O(x^{-\gamma})$ with some $\gamma > 0$.

So, we can obtain asymptotic behavior of solutions to equation (1) vanishing at $+\infty$.

Theorem 2.2. *Suppose* $2 \le n \le 4$, *$p(x) \to p_0 \neq 0$ as $x \to \infty$, $(-1)^n p_0 < 0$, and $f(x)$ satisfies condition* (4). *Then any solution $y(x)$ to equation* (1) *tending to zero as $x \to \infty$ behaves as*

$$y(x) = C\, x^{-\alpha} (1 + o(1)), \quad x \to \infty. \tag{5}$$

If $n \ge 5$ and $p(x) = p_0 + O(x^{-\gamma})$ as $x \to \infty$ with $\gamma > 0$, then there exists a solution to equation (1) *satisfying* (5).

The following theorems, which were formulated in [1]– [6], can proved similarly.

Theorem 2.3 (see [2, Ch. 2, pp. 15–16])**.** *Consider the equations*

$$y^{(2n)} + (-1)^n x^\sigma |y|^k \operatorname{sgn} y = F(x), \tag{6}$$

$$z^{(2n)} + (-1)^n x^\sigma |z|^k \operatorname{sgn} z = 0 \tag{7}$$

with $\sigma > 0$, $n \ge 1$, $k > 1$.

Suppose $|F(x)| = O(e^{-\beta x})$, $\beta > 0$, $x \to \infty$, and $y(x)$ is a solution to equation (6) *with $\lim\limits_{x \to \infty} y(x) = 0$. Then there exists a unique solution $z(x)$ to equation* (7) *such that*

$$|y(x) - z(x)| = O(e^{-\beta x}), \ \ x \to \infty.$$

Straightforward calculations show that the function $y(x) = C(x-x_0)^{-\alpha}$ with $\alpha = \frac{n}{k-1}$, $C = \big(\prod\limits_{j=0}^{n-1}(\alpha+j)\big)^{\frac{1}{k-1}}$, and arbitrary x_0 is a solution to the equation

$$y^{(n)} + (-1)^{n-1}|y(x)|^k \operatorname{sgn} y = 0, \ \ n \geq 2, \ \ k > 1. \tag{8}$$

It was proved for this equation with $n = 2$ [8] and $3 \leq n \leq 4$ [3] that all its Kneser solutions, i.e. those satisfying $y(x) \to 0$ as $x \to \infty$ and $(-1)^j y^{(j)}(x) > 0$ for $0 \leq j < n$, have the above power form. However, it was also proved [9] that for any N and $K > 1$ there exist an integer $n > N$ and $k \in (1; K)$ such that equation (1) has a solution $y(x) = (x-x_0)^{-\alpha}\, h(\log(x-x_0))$, where h is a positive periodic non-constant function on $\mathbf{R}$.

In [5] existence of that type of solutions was investigated for some fixed n.

Theorem 2.4. *Suppose $12 \leq n \leq 14$. Then there exists $k > 1$ such that equation (8) has a solution $y(x)$ satisfying*

$$y^{(j)}(x) = (x-x_0)^{-\alpha-j} h_j\big(\log(x-x_0)\big), \ \ j = 0,1,\dots,n-1,$$

with periodic positive non-constant functions h_j on $\mathbf{R}$ and arbitrary $x_0 \in \mathbf{R}$.

So, the following Theorem is proved.

Theorem 2.5. *If $12 \leq n \leq 14$, $f(x)$ satisfies (4), then there exist $k > 1$ and a solution to the equation*

$$y^{(n)} + (-1)^{n-1}|y(x)|^k \operatorname{sgn} y = F(x),$$

satisfying the condition

$$\big|y(x) - (x-x_0)^{-\alpha} h\big(\log(x-x_0)\big)\big| = O(e^{-\beta x}), \ \ x \to \infty,$$

with some periodic positive non-constant function h on $\mathbf{R}$.

2 Power-law small potential

Theorem 2.6. *Suppose the function $F(x)$ in equation (1) satisfies the condition*

$$|F(x)| \leq Cx^{-\sigma}, \ \ C > 0, \ \ \sigma > n, \tag{9}$$

and $p(x)$ is a bounded continuous function.

Then for any solution $y(x)$ to equation (1) tending to zero as $x \to \infty$ there exists a solution $z(x)$ to equation (1) with $F(x) = 0$ such that

$$|y(x) - z(x)| = O(x^{n-\sigma}), \ \ x \to \infty.$$

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