Stability and Caputo Fractional Dini Derivative of Lyapunov Functions for Caputo Fractional Differential Equations

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1 Introduction

Consider the initial value problem (IVP) for the system of fractional differential equations (FrDE) with a Caputo derivative for $0 < q < 1$,

$$ {}_{t_0}^{c}D^q x = f(t,x), \quad x(t_0) = x_0, \tag{1}$$

where $x, x_0 \in \mathbb{R}^n$, $f \in C[\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n]$, $f(t,0) \equiv 0$, $t_0 \geq 0$.

The goal of the paper is study the stability properties of zero solution of the system FrDEs (1).

The stability of fractional order systems is quite recent. There are several approaches in the literature to study stability, one of which is the Lyapunov approach. We introduce the class Λ of Lyapunov-like functions which will be used to investigate the stability of (1).

Definition 1. Let $t_0, T \in \mathbb{R}_+ :\ T > t_0$, and $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$. We will say that the function $V(t,x) : [t_0, T) \times \Delta \to \mathbb{R}_+$ belongs to the class $\Lambda([t_0,T),\Delta)$ if $V(t,x) \in C([t_0,T) \times \Delta, \mathbb{R}_+)$ is locally Lipschitzian with respect to its second argument and $V(t,0) \equiv 0$.

Results on stability in the literature via Lyapunov functions could be divided into two main groups:

- continuously differentiable Lyapunov functions (see, for example, the papers [4], [7]). Different types of stability are discussed using the Caputo derivative of Lyapunov functions which depends significantly of the unknown solution of the fractional equation. This approach requires the function to be smooth enough (at least continuously differentiable) and also some conditions involved are quite restrictive;

- continuous Lyapunov functions (see, for example, the papers [5], [6]) in which the authors use the *Dini fractional derivative* along the FrDE by

$$ {}^{c}D_{+}^{q} V(t,x) = \limsup_{h \to 0} \frac{1}{h^q} \big[V(t,x) - V(t-h, x - h^q f(t,x)) \big]. \tag{2}$$

  The "fractional Dini derivative" (2) is a strange operator since it is local and in some cases it is totally different than the used derivatives in ordinary case ($q = 1$).

Example 1. Let $V(t,x) = \frac{x^2}{(t+1)^2}$, $x \in \mathbb{R}$. Then using (2) we get $D^q V(t,x) = \frac{2xf(t,x)}{(t+1)^2}$ which is different than the used derivative in the ordinary case

$$DV(t,x) = \frac{2x}{(t+1)^2}\, f(t,x) + x^2\Big(\frac{1}{(t+1)^2}\Big)'.$$

We introduce the derivative of the Lyapunov function based on the Caputo fractional Dini derivative of a function. We define the generalized *Caputo fractional Dini derivative* of Lyapunov like function $V(t,x)$ along the system FrDE (1) by (see [1]):

$$\begin{aligned}{}^{c}_{(1)}D^q_+ V(t,x) = \limsup_{h\to 0^+} \frac{1}{h^q}\bigg\{&V(t,x) - V(t_0,x_0) - \\ &- \sum_{r=1}^{[\frac{t-t_0}{h}]} (-1)^{r+1} qCr\big[V(t-rh, x-h^q f(t,x)) - V(t_0,x_0)\big]\bigg\} \ \text{ for } \ t \geq t_0, \quad (3)\end{aligned}$$

where $t \in (t_0,T)$, $x, x_0 \in \Delta$, and there exists $h_1 > 0$ such that $t-h \in [t_0,T)$, $x - h^q f(t,x) \in \Delta$ for $0 < h \leq h_1$, $\Delta \subset \mathbb{R}^n$.

Example 2. Let $V(t,x) = \frac{x^2}{(t+1)^2}$, $x \in \mathbb{R}$ and $t_0 = 0$, $x_0 = 0$. Then using (3) we get the Caputo fractional Dini derivative ${}^{c}_{(1)}D^q_+ V(t,x) = \frac{2xf(t,x)}{(t+1)^2} + x^2 D^q_0 \frac{1}{(t+1)^2}$. which is slightly different than the ordinary case $q = 1$.

2 Comparison Results for Scalar FrDE

The base of the main results in study stability properties of FrDE (1) is the application of Caputo fractional Dini derivative (3) and some comparison results.

Lemma 1. *Assume the following conditions are satisfied:*

1. *the function $x^*(t) = x(t;t_0,x_0)$, $x^* \in C^q([t_0,T],\Delta)$ is a solution of the FrDE (1), where $\Delta \subset \mathbb{R}^n$, $0 \in \Delta$;*
2. *the function $V \in \Lambda([t_0,T],\Delta)$ and for any points $t \in [t_0,T]$, $x \in \Delta$ the inequality ${}^{c}_{(1)}D^q_+ V(t,x) \leq -c(\|x\|)$ holds, where $c \in \mathcal{K}$.*

Then for $t \in [t_0,T]$ the inequality $V(t,x^*(t)) \leq V(t_0,x_0) - \frac{1}{\Gamma(q)} \int\limits_{t_0}^{t} (t-s)^{q-1} c(\|x^*(s)\|)\, ds$ holds.

3 Stability Results

Several sufficient conditions for stability, uniform stability, asymptotic stability of zero solution of the system FrDE (1) are obtained.

Theorem 1 ([1])**.** *Assume:*
There exists a function $V \in \Lambda(\mathbb{R}_+,\mathbb{R}^n)$ such that

(i) *for any points $t \geq 0$ and $x \in \mathbb{R}^n$ the inequality ${}^{c}_{(1)}D^q_+ V(t,x) \leq -c(\|x\|)$ holds, where $c \in \mathcal{K}$;*

(ii) *$b(\|x\|) \leq V(t,x) \leq a(\|x\|)$ for $t \in \mathbb{R}_+$, $x \in \mathbb{R}^n$, where $a, b \in \mathcal{K}$.*

Then the zero solution of the FrDE (1) is uniformly asymptotically stable.

The introduced Caputo fractional Dini derivative (3) is appropriately transformed to the generalized *Caputo fractional Dini derivative w.r.t. to ITD* of the function $V(t,x)$:

$$
\begin{aligned}
{}_{t_0}^{\;c}D_{(1)}^{q}V(t,x,y,\eta,x_0,y_0) &= \lim_{h\to 0^+}\sup\frac{1}{h^q}\bigg[V(t,y-x)-V(t_0,y_0-x_0)-\\
&- \sum_{r=1}^{[\frac{t-t_0}{h}]}(-1)^{r+1}qCr\Big(V\big(t-rh,y-x-h^q(f(t+\eta,y)-f(t,x))\big)-V(t_0,y_0-x_0)\Big)\bigg], \quad (4)
\end{aligned}
$$

where $t,t_0\in I$, $y-x,y_0-x_0\in\Delta$, and there exists $h_1>0$ such that $t-h\in I$, $y-x-h^q(f(t+\eta,y)-f(t,x))\in\Delta$ for $0<h\le h_1$ and $\eta\in B_H$.

The Caputo fractional Dini derivative w.r.t. to ITD is applied to study practical stability with initial time difference for FrDE (1) (see [3]).

The base of the main results is the following result.

Lemma 2 (Shift solutions in the nonautonomous FrDE [3])**.** *Let the function $x\in C^q(\mathbb{R}_+,\mathbb{R}^n)$, $a\ge 0$, be a solution of the initial value problem for FrDE*

$$ {}_{a}^{c}D^{q}x(t)=f(t,x(t)) \ \text{ for } \ t>a, \quad x(a)=x_0. \tag{5}$$

Then the function $\tilde{x}(t)=x(t+\eta)$ satisfies the initial value problem for the FrDE

$$ {}_{b}^{c}D^{q}x=f(t+\eta,x) \ \text{ for } \ t>b, \quad x(b)=x_0, \tag{6}$$

where $b\ge 0$, $\eta=a-b$.

One of the obtained sufficient conditions are formulated below:

Theorem 2 (Uniform practical stability [3])**.** *Let the following conditions be satisfied:*

1. *The function $g\in C[[t_0,\infty)\times\mathbb{R}\times B_H,\mathbb{R}]$, $g(t,0,0)\equiv 0$ and for any parameter $\eta\in B_H$ there exists a positive number M_η such that for any $\varepsilon\in[0,M_\eta]$ and $v_0\in\mathbb{R}$ the IVP for the scalar FrDE ${}_{\tau_0}^{c}D^{q}x(t)=f(t,x(t))$, $t>\tau_0$, $x(\tau_0)=y_0$ has a solution $u(t;t_0,v_0,\eta,\varepsilon)\in C^q([t_0,\infty),\mathbb{R})$, where $H>0$ is a given number.*

2. *There exists a function $V\in\Lambda(\mathbb{R}_+,S(A))$ such that*

   (i) *$b(\|x\|)\le V(t,x)\le a(\|x\|)$ for $(t,x)\in\mathbb{R}_+\times S(A)$, where $a,b\in\mathcal{K}$;*

   (ii) *for any $t_0\in\mathbb{R}_+$, $x,y,x_0,y_0\in\mathbb{R}^n$: $y-x\in S(A)$, $y_0-x_0\in S(A)$ and $\eta\in B_H$ the inequality ${}_{t_0}^{\;c}D_{(1)}^{q}V(t,x,y,\eta,x_0,y_0)\le g(t,V(t,y-x),\eta)$ for $t\ge t_0$ holds, where $A>0$ is a given number.*

3. *The scalar FrDE ${}_{\tau_0}^{c}D^{q}x(t)=f(t,x(t))$ for $t>\tau_0$, $x(\tau_0)=y_0$ with $\varepsilon=0$ is uniformly parametrically practically stable with respect to $(a(\lambda),b(A))$, where the constant $\lambda\in(0,A)$ is given so that $a(\lambda)<b(A)$.*

Then the system of FrDE (1) *is uniformly practically stable with ITD with respect to* (λ,A) .

Also, the system of fractional differential equations with noninstantaneous impulses is defined, the Caputo fractional Dini derivative (3) is appropriately transformed and stability of the zero solution is studied in [2].

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