On One Formula of Computation of Uniform Means of Piecewise Continuous Functions on the Semiaxis

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We denote by CP a class of piecewise continuous functions $a(\,\cdot\,):[0,+\infty)\to\mathbb{R}$, and by $m(a;s,t)$ we denote an integral mean of a function $a(\,\cdot\,)\in CP$ on a segment $[s,t]$, i.e. a quantity $m(a;s,t)\overset{\text{def}}{=}(t-s)^{-1}\int_s^t a(\xi)\,d\xi$. We also denote by CPB a subclass of CP, consisting of bounded on the semiaxis functions.

In paper [1], in particular, some formulae for computation of lower $\underline{\underline{a}}$ and upper $\overline{\overline{a}}$ integral means of a function $a(\,\cdot\,)\in CP$, i.e. of quantities

$$\underline{\underline{a}}\overset{\text{def}}{=}\varliminf_{t-s\to+\infty} m(a;s,t)\ \text{ and }\ \overline{\overline{a}}\overset{\text{def}}{=}\varlimsup_{t-s\to+\infty} m(a;s,t) \tag{1}$$

are obtained. As well as a function $a(\,\cdot\,)$ belongs to CP, the values of $\underline{\underline{a}}$ and $\overline{\overline{a}}$ may be infinite ($-\infty$ or $+\infty$). All these values are obviously finite for functions $a(\,\cdot\,)\in CPB$. The general result of the paper [1] concerning computation of the quantities (1) for a function $a(\,\cdot\,)\in CP$ is that in case of their finiteness the same formulae, known for functions $a(\,\cdot\,)\in CPB$ ([2, p. 117] and [3, p. 66]), are valid. The assumption of a finite value of the quantities (1) is significant [1].

In this paper, in addition to the formulae of [1], one more formula for computation of the quantities (1) for functions of the class CP, the validity of which for functions of the class CPB was established earlier in [4] and [5], is obtained.

The properties of the quantities (1) are important in connection with the study of the lower $\underline{\beta}[x]$ and upper $\overline{\beta}[x]$ Bohl exponents [6, . 171–172; 7] of nonzero solutions $x(\,\cdot\,)$ of the linear differential system

$$\dot{x}=A(t)x,\ \ x\in\mathbb{R}^n,\ \ t\geqslant 0, \tag{2}$$

which are defined by formulae:

$$\underline{\beta}[x]=\varliminf_{t-s\to+\infty}\frac{1}{t-s}\ln\frac{\|x(t)\|}{\|x(s)\|}\ \text{ and }\ \overline{\beta}[x]=\varlimsup_{t-s\to+\infty}\frac{1}{t-s}\ln\frac{\|x(t)\|}{\|x(s)\|}\,, \tag{3}$$

and used in the Lyapunov exponent theory. In particular, choosing the function in (1) as $a(\tau)\equiv(\ln\|x(\tau)\|)'$, we obtain the quantities (3).

Following [1], for the fuction $a(\,\cdot\,)\in CP$ we denote by $T(a)$ a set of all two-dimensional sequences $((s_k,t_k))_{k\in\mathbb{N}}$ such that $t_k-s_k\to+\infty$ when $k\to+\infty$ and there exists $\lim\limits_{k\to+\infty} m(a;s_k,t_k)$, and we denote by $S(a)$ a subset of all sequences $((s_k,t_k))_{k\in\mathbb{N}}$ of $T(a)$, for which the additional condition $s_k\to+\infty$ holds. By the definitions of the lower and the upper limits, the definitions (1) of the uniform integral means $a(\,\cdot\,)$ may be written as follows

$$\underline{\underline{a}}=\inf_{((s_k,t_k))\in T(a)}\lim_{k\to+\infty} m(a;s_k,t_k)\ \text{ and }\ \overline{\overline{a}}=\sup_{((s_k,t_k))\in T(a)}\lim_{k\to+\infty} m(a;s_k,t_k). \tag{4}$$

It is, in particular, shown in [1], that the following equalities are valid

$$\underline{\underline{a}}=\inf_{((s_k,t_k))\in S(a)}\lim_{k\to+\infty} m(a;s_k,t_k)\ \text{ and }\ \overline{\overline{a}}=\sup_{((s_k,t_k))\in S(a)}\lim_{k\to+\infty} m(a;s_k,t_k). \tag{5}$$

Definitions (5), in comparison with definitions (4), constrict the class of two-dimensional sequences, which could be taken for evaluation of the limit of integral averages. The class of such sequences may be even more essentially constricted [4, 5], as it is shown below.

We fix a sequence $\delta = (\delta_k)_{k\in\mathbb{N}}$ such that $\delta_{k+1} - \delta_k \to +\infty$ when $k \to +\infty$ (every such sequence δ we will hereinafter call the rapidly increasing). We denote $\Delta_i = [\delta_i, \delta_{i+1}]$, $i \in \mathbb{N}$, and will write $s \approx t \pmod{\delta}$, or shorter $s \approx t$, if s and t belong for some i to the same segment Δ_i. It is shown in [4, 5], that for the function $a(\,\cdot\,) \in CPB$ (and this condition is essentially used in the proof) its lower and upper uniform integral means may be evaluated under the conditions $t - s \to +\infty$ and $s \approx t \pmod{\delta}$. This statement in the theorem stated below, is transferred to the class of functions CP. We denote by $S(a;\delta)$ for fixed rapidly increasing sequence δ and a function $a(\,\cdot\,) \in CP$ a subset of those sequences $((s_k,t_k))_{k\in\mathbb{N}}$ of $S(a)$, for which the condition $s_k \approx t_k$ (mod *delta*) holds.

Theorem. *For every function $a(\,\cdot\,) \in CP$ and every fixed rapidly increasing sequence δ the following equalities hold: if $\underline{\underline{a}} > -\infty$, then*

$$\underline{\underline{a}} = \varliminf_{\substack{t-s\to+\infty\\ s\approx t \pmod{\delta}}} m(a;s,t) = \inf_{((s_k,t_k))\in S(a;\delta)} \lim_{k\to+\infty} m(a;s_k,t_k), \tag{6}$$

and if $\overline{\overline{a}} < +\infty$, then

$$\overline{\overline{a}} = \varlimsup_{\substack{t-s\to+\infty\\ s\approx t \pmod{\delta}}} m(a;s,t) = \sup_{((s_k,t_k))\in S(a;\delta)} \lim_{k\to+\infty} m(a;s_k,t_k). \tag{7}$$

Let us emphasize the importance of restrictions $\underline{\underline{a}} > -\infty$ and $\overline{\overline{a}} < +\infty$ for the validity of the formulae (6) and (7), respectively. Indeed, for example, the equality (7) does not hold for the sequence $\delta = (\delta_k)_{k\in\mathbb{N}}$, where $\delta_k = k^2$, $k \in \mathbb{N}$, and the function $a(\,\cdot\,)$, given by the equalities: $a(t) = -k^2$ when $t \in [(2k-1)^2, (2k)^2 - 1)$, $a(t) = k^2$ when $t \in [(2k)^2 - 1, (2k)^2)$ and $a(t) = 1$ when $t \in [(2k)^2, (2k+1)^2)$, $k \in \mathbb{N}$.

In fact, for the so-defined function $a(\,\cdot\,)$ we have: $\overline{\overline{a}} = +\infty$, since, as is easily seen,

$$m(a;(2k)^2 - 1, (2k+1)^2) = (k^2 + 4k + 1)/(4k + 2) \to +\infty \ \text{ for } \ k \to +\infty.$$

On the other hand, the integral mean $m(a;s_i,t_i) = 1$, if $s_i, t_i \in [\delta_{2k}, \delta_{2k+1}]$, $k \in \mathbb{N}$, and $m(a;s_i,t_i) \leqslant k^2 - k^2(t_i - s_i - 1) = -k^2(t_i - s_i - 2)$, if $s_i, t_i \in [\delta_{2k-1}, \delta_{2k}]$, $k \in \mathbb{N}$, and, therefore, in this case $m(a;s_i,t_i) \leqslant 0$ when $t_i - s_i \geqslant 2$. That is why for the sequence δ and the function $a(\,\cdot\,)$ holds the equality

$$\varlimsup_{\substack{t-s\to+\infty\\ s\approx t \pmod{\delta}}} m(a;s,t) = \sup_{(s_k,t_k)\in S(a;\delta)} \lim_{k\to+\infty} m(a;s_k,t_k) = 1,$$

i.e. the first equality in (7) does not holds.

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