On a Weak Solvability of One Nonlocal Boundary-Value Problem in Weighted Sobolev Space

Givi Berikelashvili

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, Tbilisi, Georgia;
Department of Mathematics of Georgian Technical University, Tbilisi, Georgia
E-mail: bergi@rmi.ge; berikela@yahoo.com

Nodar Khomeriki

Department of Mathematics of Georgian Technical University, Tbilisi, Georgia
E-mail: n.khomeriki@gtu.ge

Let $\Omega=\{x\in\mathbb{R}^n:\ 0<x_k<1\}$ be the open unit cube in $\mathbb{R}^n$ with boundary Γ, and $\Gamma_0=\Gamma\setminus\Gamma_*$, $\Gamma_*=\{(0,x_2,\dots,x_n):\ 0<x_k<1,\ k=2,\dots,n\}$.

Consider the nonlocal boundary-value problem

$$\sum_{k=1}^n \frac{\partial}{\partial x_k}\Big(a_k\,\frac{\partial u}{\partial x_k}\Big)=-f(x),\ \ x\in\Omega, \tag{1}$$

$$u(x)=0,\ x\in\Gamma_0, \tag{2}$$

$$\ell u:=\int\limits_0^1 \beta(x_1)u(x)\,dx_1=0,\ \ 0\le x_k\le 1,\ \ k=2,\dots,n, \tag{3}$$

where $\beta(t)=\varepsilon t^{\varepsilon-1}$, $\varepsilon\in(0;1)$.

Define the operator

$$Gv=v-Hv,$$

where H is the weighted Hardy operator associated to conditions (3):

$$Hv=\frac{1}{\rho(x_1)}\int\limits_0^{x_1}\beta(t)v(t,x_2,\dots,x_n)\,dt,\quad \rho(x_1)=\int\limits_0^{x_1}\beta(t)\,dt=x_1^{\varepsilon}.$$

By $L_2(\Omega,\rho)$ we denote the weighted Lebesgue space of all real-valued functions $u(x)$ on Ω with the inner product and the norm

$$(u,v)_\rho=\int\limits_\Omega \rho(x_1)u(x)v(x)\,dx,\quad \|u\|_\rho=(u,u)_\rho^{1/2}.$$

The weighted Sobolev space $W_2^1(\Omega,\rho)$ is usually defined as a linear set of all functions $u(x)\in L_2(\Omega,\rho)$, whose distributional derivatives $\partial u/\partial x_k$, $k=1,2,\dots,n$ are in $L_2(\Omega,\rho)$. It is a normed linear space if equipped with the norm

$$\|u\|_{W_2^1(\Omega,\rho)}=\big(\|u\|_\rho^2+|u|^2_{W_2^1(\Omega,\rho)}\big)^{1/2},\quad |u|^2_{W_2^1(\Omega,\rho)}=\sum_{k=1}^n\Big\|\frac{\partial u}{\partial x_k}\Big\|_\rho^2.$$

Define the subspace of space $W_2^1(\Omega,\rho)$ which can be obtained by closing the set

$$\overset{\circ}{C}{}^\infty(\overline{\Omega})=\Big\{u\in C^\infty(\overline{\Omega}):\ \operatorname{supp}u\cap\Gamma_0=\varnothing,\ \ell u=0,\ 0<x_k<1,\ k=2,\dots,n\Big\}$$

with the norm $\|\cdot\|_{W_2^1(\Omega,\rho)}$. Denote it by $\overset{\circ}{W}{}_2^1(\Omega,\rho)$.

Let the right-hand side $f(x)$ in equation (1) be a linear continuous functional on $\overset{\circ}{W}{}_2^1(\Omega,\rho)$ which can be represented as

$$f=f_0+\sum_{k=1}^n\frac{\partial f_k}{\partial x_k},\ \ f_k(x)\in L_2(\Omega,\rho),\ \ k=0,1,2,\dots,n. \tag{4}$$

Assume

$$\begin{gathered}\nu\le a_k(x)\le\mu\ \ (k=1,\dots,n),\\ \nu,\mu=const>0,\ \ 0\le\frac{\partial}{\partial x_1}\,(a_kx_1^{1-\varepsilon})\in L_\infty(\Omega),\ \ (k=2,\dots,n).\end{gathered} \tag{5}$$

Definition. We say that the function $u\in\overset{\circ}{W}{}_2^1(\Omega,\rho)$ is a weak solution of problem (1)–(5) if the relation

$$a(u,v)=\langle f,v\rangle,\ \ \forall\,v\in\overset{\circ}{W}{}_2^1(\Omega,\rho) \tag{6}$$

holds, where

$$\begin{gathered}a(u,v)=\Big(a_1\,\frac{\partial u}{\partial x_1},\frac{\partial v}{\partial x_1}\Big)_\rho+\sum_{k=2}^n\Big(a_k\,\frac{\partial u}{\partial x_k},G\,\frac{\partial v}{\partial x_k}\Big)_\rho,\\ \langle f,v\rangle=(f_0,Gv)_\rho-\sum_{k=1}^n\Big(f_k,\frac{\partial}{\partial x_k}\,Gv\Big)_\rho.\end{gathered}$$

Equality (6) formally is obtained from $(\Delta u+f,Gv)_\rho=0$ by integration by parts, taking into account that

$$\begin{aligned}\Big(\frac{\partial v}{\partial x_1},Gu\Big)_\rho&=-\Big(v,\frac{\partial u}{\partial x_1}\Big)_\rho,\\ \Big(\frac{\partial v}{\partial x_k},Gu\Big)_\rho&=-\Big(v,G\,\frac{\partial u}{\partial x_k}\Big)_\rho,\ \ k=2,\dots,n.\end{aligned}$$

Theorem. *The problem* (1)–(5) *has a unique weak solution from* $\overset{\circ}{W}{}_2^1(\Omega,\rho)$.