On the Well-Possed Question of the Antiperiodic Problem for Systems of Linear Generalized Differential Equations

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We consider the well-possed question for the $\omega$-antiperiodic problem for linear generalized ordinary differential equations of the form

$$
dx(t) = dA(t) \cdot x(t) + df(t) \quad t \in \mathbb{R}$$

under the $\omega$-antiperiodic condition

$$x(t + \omega) = -x(t) \quad t \in \mathbb{R},$$

where $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ and $f : \mathbb{R} \to \mathbb{R}^n$ are, respectively, matrix- and vector-functions with bounded variation components on every closed interval from $\mathbb{R}$, and $\omega$ is a fixed positive number.

Let the system (1) have the unique $\omega$-antiperiodic solution $x_0$.

Along with the system (1) consider the sequence of the systems

$$
dx(t) = dA_k(t) \cdot x(t) + df_k(t) \quad (k = 1, 2, \ldots),$$

where $A_k : \mathbb{R} \to \mathbb{R}^{n \times n}$ and $f_k : \mathbb{R} \to \mathbb{R}^n$ are, respectively, matrix- and vector-functions with bounded variation components on every closed interval from $\mathbb{R}$.

We give the necessary and sufficient condition for a sequence of $\omega$-antiperiodic problems (1_k), (2) $(k = 1, 2, \ldots)$ to have a unique solution $x_k$ for sufficiently large $k$ and

$$\lim_{k \to +\infty} x_k(t) = x_0(t)$$

uniformly on $\mathbb{R}$.

To a considerable extent, the interest to the theory of generalized ordinary differential equations has also been stimulated by the fact that this theory enables one to investigate ordinary differential, impulsive and difference equations from a unified point of view (see, [1], [2], [4], [5] and references therein).

The theory of generalized ordinary differential equations has been introduced by J. Kurzweil in connection with investigation the well-possed problem for the Cauchy problem for ordinary differential equations.

The use will be made of the following notation and definitions.

$\mathbb{R}$ is the real axis, $\mathbb{R}^{n \times m}$ is the space of all real $n \times m$ matrices. $O_{n \times m}$ is the zero $n \times m$ matrix. $I_n$ is the identity $n \times n$-matrix. $\mathbb{R}^n = \mathbb{R}^{n \times 1}$ is the space of all real column $n$-vectors.

If $X : [a, b] \to \mathbb{R}^{n \times m}$ is a matrix-function, then $\nabla(X)$ is the sum of total variations on $[a, b]$ of its components $x_{ij}$ $(i = 1, \ldots, n; j = 1, \ldots, m)$; $V(X)(t) = (V(x_{ij})(t))_{i,j=1}^{n,m}$, where $V(x_{ij})(a) = 0$, $V(x_{ij})(t) = \frac{t}{a} x_{ij}$ for $a < t \leq b$; $X(t-)$ and $X(t+)$ are, respectively, the left and the right limits of $X$ at the point $t$.

$BV([a, b], \mathbb{R}^{n \times m})$ is the normed space of all bounded variation matrix-functions $X : [a, b] \to \mathbb{R}^{n \times m}$ (i.e. $\int_a^b (X) < \infty$) with the norm $\|X\|_s = \sup\{\|X(t)\| : t \in [a, b]\}$.
\$\text{BV}_{loc}(\mathbb{R}, \mathbb{R}^{n \times m})$ is the space of all matrix-functions $X : [a, b] \rightarrow \mathbb{R}^{n \times m}$ whose restrictions on every closed interval $[a, b]$ from $\mathbb{R}$ belong to $\text{BV}([a, b], \mathbb{R}^{n \times n})$.

$\text{BV}_+^\omega(\mathbb{R}, \mathbb{R}^{n \times m})$ and $\text{BV}_-^\omega(\mathbb{R}, \mathbb{R}^{n \times m})$ are the sets of all matrix-functions $G : \mathbb{R} \rightarrow \mathbb{R}^{n \times m}$ whose restrictions on $[0, \omega]$ belong to $\text{BV}([0, \omega], \mathbb{R}^{n \times m})$ and there exists a constant matrix $C \in \mathbb{R}^{n \times m}$ such that, respectively, $G(t+\omega) \equiv G(t) + C$ and $G(t+\omega) \equiv G(t) - C$.

$s_j : \text{BV}([a, b], \mathbb{R}) \rightarrow \text{BV}([a, b], \mathbb{R})$ $(j = 0, 1, 2)$ are the operators defined, respectively, by $s_1(x)(a) = s_2(x)(a) = 0$, $s_1(x)(t) = \sum_{a < t \leq s} d_1x(\tau)$ and $s_2(x)(t) = \sum_{a < t \leq s} d_2x(\tau)$ for $a < t \leq b$, and $s_0(x)(t) \equiv x(t) - s_1(x)(t) - s_2(x)(t)$.

If $g : [a, b] \rightarrow \mathbb{R}$ is a nondecreasing function, $x : [a, b] \rightarrow \mathbb{R}$ and $s < t$, then $\int_a^b x(\tau) d\mu(\tau) = \int_{[s,t]} x(\tau) ds_c(g)(\tau) + \sum_{s < \tau \leq t} x(\tau) d\mu(\tau)$, where $\int_{[s,t]} x(\tau) ds_c(g)(\tau)$ is the Lebesgue–Stieltjes integral over the open interval $[s, t[ \,$ with respect to the measure $\mu_0(s_c(g))$ corresponding to the function $s_c(g)$. So that $\int_{[s,t]} x(\tau) d\mu(\tau)$ is the Kurzweil–Stieltjes integral (see, [3]–[5]).

We use the operators. If $X \in \text{BV}_{loc}(\mathbb{R}; \mathbb{R}^{n \times m})$ and $Y \in \text{BV}_{loc}(\mathbb{R}; \mathbb{R}^{n \times m})$, then $B(X, Y)(t) = X(t)Y(t) - X(0)Y(0) - \int_0^t dX(\tau) \cdot Y(\tau); \text{if, in addition, } \det(X(t)) \neq 0$ for $t \in \mathbb{R}$, then $I(X, Y)(t) = \int_0^t d(X(\tau) + B(X, Y)(\tau)) \cdot X^{-1}(\tau)$.

A vector-function $\text{BV}_{loc}(\mathbb{R}, \mathbb{R}^{n \times m})$ is said to be solution of the system (1) if $x(t) - x(s) = \int_s^t A(\tau) \cdot x(\tau) + f(\tau) - f(s) \text{ for } s < t, \, s, t \in \mathbb{R}$.

We will assume that $A, A_k \in \text{BV}_+^\omega(\mathbb{R}, \mathbb{R}^{n \times n})$ and $f, f_k \in \text{BV}_-^\omega(\mathbb{R}, \mathbb{R}^n)$ $(k = 1, 2, \ldots)$, i.e. $A(t + \omega) = A(t) + C$, $A_k(t + \omega) = A_k(t) + C_k$ and $f(t + \omega) = f(t) + C_k$ $(k = 1, 2, \ldots)$ where $C, C_k \in \mathbb{R}^{n \times n}$ $(k = 1, 2, \ldots)$ and $c, c_k \in \mathbb{R}^n$ $(k = 1, 2, \ldots)$ are, respectively, some constant matrices and vectors. In addition, without loss of generality we assume that $A(0) = A_k(0) = 0_{n \times n}$ and $f(0) = f_k(0) = 0$ $(k = 1, 2, \ldots)$. Moreover, we assume $\det(I_n + (-1)^{j} d_j A(t)) \neq 0$ for $t \in \mathbb{R}$ $(j = 1, 2)$.

**Definition 1.** We say that a sequence $(A_k, f_k)$ $(k = 1, 2, \ldots)$ belongs to the set $\mathcal{S}(A, f)$ if the $\omega$-antiperiodic problem (1), (2) has a unique solution $x_k$ for any sufficiently large $k$ and the condition (3) holds.

**Statement 1.** The following statements are valid:

(a) if $x$ is a solution of the system (1), then the function $y(t) = -x(t + \omega)$ $(t \in \mathbb{R})$ is a solution of the system (1), as well;

(b) the problem (1), (2) is solvable if and only if the system (1) on the closed interval $[0, \omega]$ has a solution satisfying the boundary condition

$$x(0) = -x(\omega).$$

More than, the set of restrictions of the solutions of the problem (1), (2) on $[0, \omega]$ coincides with the set of solutions of the problem (1), (4).

**Theorem 1.** The inclusion

$$\left((A_k, f_k)\right)_{k=1}^\infty \subset \mathcal{S}(A, f)$$

is valid if and only if there exists a sequence of matrix-functions $H_k \in \text{BV}([0, \omega], \mathbb{R}^{n \times n})$ $(k = 1, 2, \ldots)$ such that

$$\lim_{k \to +\infty} \sup_0^\omega (H_k + B(H_k, A_k)) < +\infty,$$
such that $h$ has a unique $H$ function $H$!

The conditions

$$
\lim_{k \to +\infty} H_k(t) = H(t),
$$

are fulfilled uniformly on $[0, \omega]$.

**Theorem 2.** Let $A_k \in BV([0, \omega], \mathbb{R}^{n \times n})$, $f_s \in BV([0, \omega], \mathbb{R}^n)$ be such that $\det (I_n + (-1)^j d_j A_s(t)) \neq 0$ for $t \in [0, \omega]$ ($j = 1, 2$)

and the system

$$
dx(t) = dA_s(t) \cdot x(t) + df_s(t)
$$

has a unique $\omega$-antiperiodic solution $x_s$. Let, moreover, there exist sequences of matrix-and vector-functions $H_k \in BV([0, \omega], \mathbb{R}^{n \times n})$ ($k = 1, 2, \ldots$) and $h_k \in BV([0, \omega], \mathbb{R}^n)$ ($k = 1, 2, \ldots$), respectively, such that $h_k(0) = -h_k(\omega)$ ($k = 1, 2, \ldots$), inf $\{\det(H_k(t)) : t \in [0, \omega]\} > 0$ ($k = 1, 2, \ldots$),

$$
l\lim k \sup_{k \to +\infty} b A_{sk} < +\infty, \text{ and the conditions } \lim_{k \to +\infty} A_{sk}(t) = A_s(t) \text{ and } \lim_{k \to +\infty} f_{sk}(t) = f_s(t)
$$

are fulfilled uniformly on $[0, \omega]$, where $A_{sk}(t) \equiv I_k(H_k, A_k)(t) (k = 1, 2, \ldots)$ and

$$
f_{sk}(t) = h_k(t) - h_k(0) + B_k(H_k, f_k)(t) - \int_0^t dA_{sk}(\tau) \cdot h_k(\tau) (k = 1, 2, \ldots).
$$

Then the system $\{I_k\}$ has the unique $\omega$-antiperiodic solution $x_k$ for any sufficiently large $k$ and

$$
\lim_{k \to +\infty} \|H_k x_k + h_k - x_s\| = 0.
$$

**Corollary 1.** Let the conditions (6) and (7) hold, and let the conditions (8), (9) and

$$
\lim_{k \to +\infty} \left( B(H_k, f - \varphi_s(t) + \int_0^t dB(H_k, A_k)(s) \cdot \varphi_k(s) \right) = B(H, f)(t)
$$

be fulfilled uniformly on $[0, \omega]$, where $H, H_k \in BV([0, \omega], \mathbb{R}^{n \times n})$ ($k = 1, 2, \ldots$). Then the system $\{I_k\}$ has a unique $\omega$-antiperiodic solution $x_k$ for any sufficiently large $k$ and

$$
\lim_{k \to +\infty} \|x_k - \varphi_s - x_s\| = 0.
$$

**Corollary 2.** Let the conditions (6) and (7) hold, and let the conditions (8),

$$
\lim_{k \to +\infty} \int_0^t H_k(s) dA_k(s) = \int_0^t H(s) dA(s), \lim_{k \to +\infty} \int_0^t H_k(s) df_k(s) = \int_0^t H(s) df(s),
$$

be fulfilled uniformly on $[0, \omega]$, where $H, H_k \in BV([0, \omega], \mathbb{R}^{n \times n})$ ($k = 1, 2, \ldots$). Let, moreover, either

$$
\lim_{k \to +\infty} \sup_{a \leq t \leq b} \left( \|d_j A_k(t)\| + \|d_j f_k(t)\| \right) < +\infty (j = 1, 2),
$$

or

$$
\lim_{k \to +\infty} \sup_{a \leq t \leq b} \|d_j H_k(t)\| < +\infty (j = 1, 2).
$$

Then the inclusion (5) holds.
Corollary 3. Let the conditions (6) and (7) hold, and let the conditions (8),

\[
\lim_{k \to +\infty} A_k(t) = A(t),
\]

\[
\lim_{k \to +\infty} f_k(t) = f(t),
\]

\[
\lim_{k \to +\infty} \int_0^t d(H^{-1}(s)H_k(s)) \cdot A_k(s) = A_\ast(t), \quad \lim_{k \to +\infty} \int_0^t d(H^{-1}(s)H_k(s)) \cdot f_k(s) = f_\ast(t)
\]

be fulfilled uniformly on \([0, \omega]\), where \(H, H_k, A_\ast \in \text{BV}([0, \omega], \mathbb{R}^{n \times n})\) \((k = 1, 2, \ldots)\), and \(f_\ast \in \text{BV}([0, \omega], \mathbb{R}^n)\). Let, moreover, the system

\[
dx(t) = d(A(t) - A_\ast(t)) \cdot x(t) + d(f(t) - f_\ast(t))
\]

has the unique \(\omega\)-antiperiodic solution. Then \(((A_k, f_k))_{k=1}^{+\infty} \in S(A - A_\ast, f - f_\ast)\).

Corollary 4. Let there exist a natural number \(m\) and matrix-functions \(B_j \in \text{BV}([0, \omega], \mathbb{R}^{n \times n})\) \((j = 0, \ldots, m-1)\) such that \(\lim_{k \to +\infty} \sup_0^{\omega} (A_{km}(s)) < +\infty\), and the conditions \(\lim_{k \to +\infty} (A_{km}(t) - A_{km}(0)) = A(t)\) and \(\lim_{k \to +\infty} (f_{km}(t) - f_{km}(0)) = f(t)\), be fulfilled uniformly on \([0, \omega]\), where

\[
H_{k0}(t) \equiv I_n, \quad H_{k,j+10}(t) \equiv \prod_{j+1}^1 (I_n - A_{kl}(t) + A_{kl}(0) + B_l(t) - B_l(0)),
\]

\[
A_{k,j+1} \equiv H_{kj}(t) + B(H_{kj}, A_k)(t), \quad f_{k,j+1} \equiv B(H_{kj}, f_k)(t).
\]

Then the inclusion (5) holds.

If \(m = 1\), then Corollary 4 has the following form.

Corollary 5. Let \(\lim_{k \to +\infty} \sup_0^{\omega} (A_k) < +\infty\), and the conditions (10) and (11) be fulfilled uniformly on \([0, \omega]\). Then the inclusion (5) holds.

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References


