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THE THIRD AND FOURTH ORDER ACCURACY DECOMPOSITION FORMULAS FOR A SEMIGROUP

The first decomposition formula for an exponential matrix function was constructed by Lie in 1875. Namely, he showed that

$$\lim_{n\to\infty}\big(\exp(A_1/n)\exp(A_2/n)\big)^n=\exp(A_1+A_2),$$

where A_1 and A_2 are finite dimensional matrices. In 1959, Trotter generalized this formula for exponential operator function − semigroup $\exp(t(A_1+A_2))$ (A_1 and A_2 are self-adjoined positively defined operators). Resolvent analog to this formula for the first time was constructed by Chernoff in 1968.

Note that Lie's formula is of the first order accuracy decomposition formula, that is: $(\exp(A_1/n)\exp(A_2/n))^n-\exp(A_1+A_2)=Op(1/n)$, where the norm of the matrix $Op(1/n)$ is of the $O(1/n)$ order. There are also known the second order accuracy decomposition formulas. The third and fourth order accuracy decomposition formulas for two-dimensional splitted operator was constructed by B. O. Dia and M. Schatzman in 1996. Note that the formulas constructed by these authors are not automatically stable decomposition formulas. Decomposition formula is called automatically stable if the sum of absolute values of the split coefficients is equal to one. Q. Sheng has proved that in the real number field there does not exist such automatically stable splitting of the exponential operator function-semigroup whose accuracy order is higher than two.

We introduce the notation: $T_1(\lambda)=\exp(-\tau\lambda A_1)\cdots\exp(-\tau\lambda A_m)$ and $T_2(\lambda)=\exp(-\tau\lambda A_m)\cdots\exp(-\tau\lambda A_1)$, where λ is a scalar, and $A_1,\dots,A_m$ ($m\geq 2$) are linear (generally unbounded) operators; $A=A_1+\cdots+A_m$, $\tau=t/n$ ($t\geq 0$, n − is a natural number). The following formulas are true:

$$\exp(-tA)-\Big(\frac{1}{2}\big(T_1(\alpha)T_2(\overline{\alpha})+T_2(\alpha)T_1(\overline{\alpha})\big)\Big)^n=Op(\tau^3),$$

$$\begin{aligned}\exp(-tA)-\Bigg[\frac{1}{2}\Big(&T_1\Big(\frac{\alpha}{2}\Big)T_2\Big(\frac{\overline{\alpha}}{2}\Big)T_1\Big(\frac{\overline{\alpha}}{2}\Big)T_2\Big(\frac{\alpha}{2}\Big)+\\&+T_2\Big(\frac{\alpha}{2}\Big)T_1\Big(\frac{\overline{\alpha}}{2}\Big)T_2\Big(\frac{\overline{\alpha}}{2}\Big)T_1\Big(\frac{\alpha}{2}\Big)\Big)\Bigg]^n=Op(\tau^4),\end{aligned}$$

where $\alpha=\frac{1}{2}\pm i\frac{1}{2\sqrt{3}}$; operators $(-\alpha A_j)$, $(-\overline{\alpha}A_j)$ ($j=1,\dots,m$) and $(-A)$ generate strongly continuous semigroups; $Op(\tau^k)$ is an operator whose norm is of the fourth order with respect to τ (more precisely, in case of unbounded operators $\|Op(\tau^k)\varphi\|=O(\tau^k)$ for any φ from the domain of definition of the operator $Op(\tau^k)$).

Note that the decomposition formulas constructed by us are automatically stable.