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AN EXPONENTIAL MARTINGALE EQUATION

Let $(\Omega, \mathcal{F}, P)$ be a probability space with continuous filtration $F = (F_t, t \in [0,T])$. Denote by $\mathcal{M}$ a stable subspace of the space of square integrable martingales and let $\mathcal{M}^\perp$ be its strongly orthogonal complement.

We consider the following exponential equation

$$\mathcal{E}_T(m)\mathcal{E}^\alpha_T(m^\perp) = c\exp\{\eta\}, \tag{1}$$

where η is a given F_T-measurable random variable and α is a given real number. A solution of equation (1) is a triple $(c, m, m^\perp)$, where c is strictly positive constant, $m \in \mathcal{M}$ and $m^\perp \in \mathcal{M}^\perp$. Here $\mathcal{E}(X)$ is the Doleans–Dade exponential of X.

Equations of such type arose in mathematical finance and they are used to characterize optimal martingale measures.

Our aim is to prove the existence of a unique solution of equation (1) for arbitrary $\alpha \neq 0$ and η of a general structure, assuming that it satisfies the following boundedness condition:

A) η is an F_T-measurable random variable of the form $\eta = \xi + \gamma A_T$, where $\xi \in L^\infty$, γ is a constant and $A = (A_t, t \in [0,T])$ is a continuous F-adapted increasing process such that $E(A_T - A_\tau / F_\tau) \le C$ for all stopping times τ for a constant $C > 0$.

Note that if $\alpha \neq 0$ (if $\alpha = 0$, solution of (1) does not exist in general), equation (1) is equivalent to the following semimartingale backward equation with the square generator

$$Y_t = Y_0 - \frac{\gamma}{2} A_t - \langle L\rangle_t - \frac{1}{\alpha}\langle L^\perp\rangle_t + L_t + L^\perp_t, \quad Y_T = \frac{1}{2}\xi. \tag{2}$$

Theorem 1. *Let $\alpha \neq 0$ and condition* A) *be satisfied. Then there is a constant $\gamma_0 > 0$ such that for any $|\gamma| \le \gamma_0$ there exists a unique triple $(c, m, m^\perp)$, where $c \in R_+$, $m \in BMO \cap \mathcal{M}$, $m^\perp \in BMO \cap \mathcal{M}^\perp$, that satisfies equation* (1).

If the filtration generated by 2-dimensional Wiener processs $(w, w^\perp)$ and $\xi = 2g(w_T, w^\perp_T)$, $A_t = \int\limits_0^t f(s, w_s, w^\perp_s)ds$ for some continuous, bounded functions $g(x,y), f(s,x,y)$, equation (2) is equivalent to the PDE

$$V_t + \frac{1}{2}V_{xx} + \frac{1}{2}V_{yy} + |V_x|^2 + \frac{1}{\alpha}|V_y|^2 + \frac{\gamma}{2}f(t,x,y) = 0, \quad V(T,x,y) = g(x,y) \tag{3}$$

and solutions of (2) and (3) are related as $Y_t = V(t, w_t, w^\perp_t)$.